

Exercises to Relativistic Quantum Field Theory — Sheet 10

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Exercise 10.1 *Solutions of the free Dirac equation* (1 point)

Consider the solutions of the free Dirac equation in momentum space, $u_\sigma(k)$ and $v_\sigma(k)$ with $\sigma = 1, 2$, which are normalised according to

$$\bar{u}_\sigma(k)u_{\sigma'}(k) = -\bar{v}_\sigma(k)v_{\sigma'}(k) = 2m \delta_{\sigma\sigma'}, \quad \bar{u}_\sigma(k)v_{\sigma'}(k) = \bar{v}_\sigma(k)u_{\sigma'}(k) = 0.$$

a) Prove the completeness relation

$$\sum_{\sigma=1,2} \left(u_\sigma(k) \otimes \bar{u}_\sigma(k) - v_\sigma(k) \otimes \bar{v}_\sigma(k) \right) = 2m \mathbf{1}.$$

b) Express the matrices

$$\Lambda_+(k) = \frac{1}{2m} \sum_{\sigma=1,2} u_\sigma(k) \otimes \bar{u}_\sigma(k), \quad \Lambda_-(k) = -\frac{1}{2m} \sum_{\sigma=1,2} v_\sigma(k) \otimes \bar{v}_\sigma(k)$$

in terms of m , \not{k} , and the identity matrix. Argue that $\Lambda_\pm(k)$ are orthogonal projectors onto the subspaces of positive and negative energies, respectively.

c) Show that

$$\bar{u}_\sigma(k)\gamma_\mu u_{\sigma'}(k) = \bar{v}_\sigma(k)\gamma_\mu v_{\sigma'}(k) = 2k_\mu \delta_{\sigma\sigma'}$$

by evaluating $\bar{u}_\sigma(k)\{\gamma_\mu, \not{k}\}u_{\sigma'}(k)$ in two different ways.

d) Similarly, show that

$$v_\sigma(\tilde{k})^\dagger u_{\sigma'}(k) = u_\sigma(\tilde{k})^\dagger v_{\sigma'}(k) = 0,$$

using $\gamma_0 \not{k} = \tilde{k} \gamma_0$, where $(\tilde{k}^\mu) = (k_0, -\vec{k})$ for $(k^\mu) = (k_0, \vec{k})$.

Exercise 10.2 *Polarisation sums for Dirac spinors* (0.5 points)

As in Exercise 10.1, $u_\sigma(p)$ and $v_\tau(P)$ with $\sigma, \tau \in \{1, 2\}$ denote the solutions of the free Dirac equation in momentum space, where $p^2 = m^2$, $P^2 = M^2$. Show that the following polarisation sum $\sum_{\sigma, \tau}$ can be written as a trace in Dirac space according to

$$\sum_{\sigma, \tau} \left(\bar{u}_\sigma(p) \Gamma v_\tau(P) \right)^* \left(\bar{u}_\sigma(p) \Gamma v_\tau(P) \right) = \text{Tr} \left[(\not{P} - M) \tilde{\Gamma} (\not{p} + m) \Gamma \right],$$

where Γ is an arbitrary 4×4 matrix and $\tilde{\Gamma} = \gamma_0 \Gamma^\dagger \gamma_0$.

Please turn over!

Exercise 10.3 *Field operator of the free Dirac fermion* (1 point)

Consider the following plane-wave expansion of the field operator $\psi(x)$ of the free Dirac fermion,

$$\psi(x) = \int d\tilde{p} \sum_{\sigma} \left(e^{-ipx} u_{\sigma}(p) a_{\sigma}(\vec{p}) + e^{+ipx} v_{\sigma}(p) b_{\sigma}^{\dagger}(\vec{p}) \right),$$

where $a_{\sigma}^{(\dagger)}(\vec{p})$ and $b_{\sigma}^{(\dagger)}(\vec{p})$ denote the annihilation (creation) operators of the particle and antiparticle states, respectively. The creation and annihilation operators obey the anti-commutation relations

$$\{a_{\sigma}(\vec{p}), a_{\tau}^{\dagger}(\vec{k})\} = \{b_{\sigma}(\vec{p}), b_{\tau}^{\dagger}(\vec{k})\} = 2p_0 (2\pi)^3 \delta_{\sigma\tau} \delta^{(3)}(\vec{p} - \vec{k}),$$

and all other anticommutators vanish.

- a) Calculate the operator

$$\hat{Q} = Qe \int d^3x : \bar{\psi}(x) \gamma_0 \psi(x) :$$

of electric charge in terms of an integral in momentum space.

- b) Calculate the commutators $[\hat{Q}, a_{\sigma}^{(\dagger)}(\vec{p})]$ and $[\hat{Q}, b_{\sigma}^{(\dagger)}(\vec{p})]$ and interpret the result.

Exercise 10.4 *Free Dirac propagator* (1 bonus point)

The free Dirac propagator is given by

$$S_F(x, y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{\not{k} + m}{k^2 - m^2 + i\epsilon}.$$

- a) Upon carrying out the k_0 -integration, show that $S_F(x, y)$ can be written as

$$\begin{aligned} S_F(x, y) &= -i\theta(x_0 - y_0) \int d\tilde{k} e^{-ik(x-y)} (m + \not{k}) \\ &\quad - i\theta(y_0 - x_0) \int d\tilde{k} e^{+ik(x-y)} (m - \not{k}). \end{aligned}$$

- b) We denote the solutions of the free Dirac equation with positive and negative energies $\psi^{(+)}(x)$ and $\psi^{(-)}(x)$, respectively. Show that

$$\begin{aligned} \theta(x_0 - y_0) \psi^{(+)}(x) &= i \int d^3y S_F(x, y) \gamma_0 \psi^{(+)}(y), \\ \theta(y_0 - x_0) \psi^{(-)}(x) &= -i \int d^3y S_F(x, y) \gamma_0 \psi^{(-)}(y). \end{aligned}$$