Exercises to Relativistic Quantum Field Theory — Sheet 6
Prof. S. Dittmaier, Universität Freiburg, WS 2019/20

Exercise 6.1 *Momentum of the quantised free scalar field (cont'd)* (1.5 points)

a) How does the energy-momentum operator P^μ act on a state $a^\dagger(\vec{p}_1)\dots a^\dagger(\vec{p}_n)|0\rangle$?

b) The action of an operator A on a wave function $\varphi(t,\vec{x}) = \langle 0|\phi(t,\vec{x})|\varphi\rangle$ corresponding to a state $|\varphi\rangle$ is defined by $A\varphi(t,\vec{x}) = \langle 0|\phi(t,\vec{x})A|\varphi\rangle$. Determine how P^μ acts on a one-particle wave function $\varphi_{\vec{p}}(t,\vec{x}) = \langle 0|\phi(t,\vec{x})|\vec{p}\rangle$ with $|\vec{p}\rangle = a^\dagger(\vec{p})|0\rangle$ in two different ways and use the results to calculate $\varphi_{\vec{p}}(t,\vec{x})$. Do not use the representation of the field operator $\phi(t,\vec{x})$ in terms of creation and annihilation operators here.

c) Express the one-particle wave function $\varphi_f(t,\vec{x})$ corresponding to the state

$$|f\rangle = \int \mathrm{d}\tilde{p}\, f(\vec{p})|\vec{p}\rangle$$

in terms of $\varphi_{\vec{p}}(t,\vec{x})$. Here $f(\vec{p})$ denotes a square-integrable "spectral function". Show that $\varphi_f(t,\vec{x})$ satisfies the Klein-Gordon equation.

Exercise 6.2 *Identities of the scalar field operator* (1 point)

Consider the field operator $\phi(x)$ of the free, real Klein-Gordon field.

a) Show that

$$[\phi(x),\phi(y)] = \int \mathrm{d}\tilde{k}\left(\mathrm{e}^{-\mathrm{i}k(x-y)} - \mathrm{e}^{+\mathrm{i}k(x-y)}\right)$$

and argue why $[\phi(x),\phi(y)] = 0$ for $(x-y)^2 < 0$, as demanded by causality.

b) Prove the relation between time ordering and normal ordering:

$$\begin{aligned} T\{\phi(x)\phi(y)\} &\equiv \theta(x^0-y^0)\phi(x)\phi(y) + \theta(y^0-x^0)\phi(y)\phi(x) \\ &= :\phi(x)\phi(y): + \langle 0|T\{\phi(x)\phi(y)\}|0\rangle. \end{aligned}$$

Exercise 6.3 *Normalisation of multi-particle states* (1 point)

Show that the n-particle states in (bosonic) Fock space,

$$|\vec{p}_1,\dots\vec{p}_n\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)\dots a^\dagger(\vec{p}_n)|0\rangle,$$

are normalised according to

$$\langle\vec{p}_1,\dots\vec{p}_n|\vec{k}_1,\dots,\vec{k}_m\rangle = \delta_{mn}\,(2\pi)^{3n}\sum_{\pi\in S_n}\prod_{i=1}^{n}(2p_i^0)\,\delta^{(3)}(\vec{p}_i - \vec{k}_{\pi(i)}),$$

where the sum runs over all permutations S_n of the indices $\{1,\dots,n\}$.