Exercises to Relativistic Quantum Field Theory — Sheet 4
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Exercise 4.1 *Non-relativistic propagator* (2 points)

In Exercise 3.1 the Green's function of Schrödinger's equation was introduced as the solution of the differential equation

$$\left(\mathrm{i}\frac{\partial}{\partial t} + \frac{1}{2m}\Delta - V(\vec{x})\right) G_(t,\vec{x};t',\vec{x}\,') = \delta(t-t')\delta^{(3)}(\vec{x}-\vec{x}\,'). \tag{1}$$

a) Determine the Fourier transform of the Green's function G_0 of the free Schrödinger equation with $V(\vec{x}) = 0$, i.e. write G_0 in the form

$$G_0(t,\vec{x};t',\vec{x}\,') = \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\omega(t-t')}\mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{x}\,')}\tilde{G}_0(\omega,\vec{p}) \tag{2}$$

and determine the function $\tilde{G}_0(\omega,\vec{p})$. Why can the free Green's function only depend on $t-t'$ and $\vec{x}-\vec{x}\,'$?

$\left[\textit{Result:}\quad \tilde{G}(\omega,\vec{p}) = \left(\omega - \frac{\vec{p}^{\,2}}{2m}\right)^{-1}.\right]$

b) Now perform the ω integration in (2) after shifting the pole in ω according to

$$G_0^{(\pm)}(t,\vec{x}) = \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}(\omega t-\vec{p}\cdot\vec{x})}\, \frac{1}{\omega - \frac{\vec{p}^{\,2}}{2m} \pm \mathrm{i}\epsilon}. \tag{3}$$

with an infinitesimal $\epsilon > 0$. Which signs of $\pm\mathrm{i}\epsilon$ correspond to the retarded and advanced cases?

Hint: Prove first and then use $\theta(\pm\tau) = \frac{\mp 1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{\omega \pm \mathrm{i}\epsilon}$.

c) Calculate $G_0^{(\pm)}(t,\vec{x})$ explicitly upon carrying out the integration over d^3p, starting from the result of b).

Hint: Use the auxiliary integral $\int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{-a(z+b)^2} = \sqrt{\pi/a}$, where $a, b \in \mathbb{C}$, $a \neq 0$, $\mathrm{Re}(a) \geq 0$.

Please turn over!

Exercise 4.2 *Electromagnetic interaction of charged scalars* (2.5 points)

Generically the interaction of the electromagnetic field $(A^\mu) = (\phi, \vec{A})$ with charged fields is described by a Lagrangian of the form

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu + \mathcal{L}_0,$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field-strength tensor, $(j^\mu) = (\rho, \vec{j})$ the four-current density of the charges, and $\mathcal{L}_0$ the Lagrangian for the free propagation of the charges (and possibly other interactions among them), i.e. $\mathcal{L}_0$ does not depend on A^μ.

Comment: In relativistic field theory it is customary to use Lorentz-Heaviside units, which result from the SI units upon setting $\mu_0 = \varepsilon_0 = c = 1$.

a) Express the electromagnetic field-strength tensor $F^{\mu\nu}$ and its dual $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ in terms of the electric field $\vec{E} = -\nabla\phi - \dot{\vec{A}}$ and the magnetic flux density $\vec{B} = \nabla \times \vec{A}$.

b) Bring the homogeneous Maxwell equations $\partial_\mu \tilde{F}^{\mu\nu} = 0$ into their usual form in terms of $\vec{E}$ and $\vec{B}$. Show that $\partial_\mu \tilde{F}^{\mu\nu} = 0$ follows from the definitions of $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$.

c) Derive the inhomogeneous Maxwell equations for the field strength in their covariant form $\partial_\mu F^{\mu\nu} = j^\nu$ from the Euler-Lagrange equations for A^μ and bring them into their usual form in terms of $\vec{E}$ and $\vec{B}$. Verify current conservation $\partial_\mu j^\mu = 0$.

d) Now consider a complex scalar field Φ to describe a spinless particle with electric charge q and mass m, as in Exercise 3.3b). The free propagation of Φ is described by

$$\mathcal{L}_0(\Phi, \partial\Phi) = (\partial\Phi)^*(\partial\Phi) - m^2\Phi^*\Phi.$$

The electromagnetic interaction between Φ and A^μ is introduced by the "minimal substitution" $\partial^\mu \to D^\mu = \partial^\mu + iqA^\mu$ in $\mathcal{L}_0$, resulting in

$$\mathcal{L}_\Phi(\Phi, \partial\Phi, A) = \mathcal{L}_0(\Phi, D\Phi) = \mathcal{L}_0(\Phi, \partial\Phi) - j_\mu A^\mu.$$

Derive the explicit form of the current density j^μ.

e) $\mathcal{L}_\Phi$ is invariant under the global transformation $\Phi \to \Phi' = \exp(-\mathrm{i}q\omega)\Phi$, with ω denoting an arbitrary real number. Derive the Noether current corresponding to this symmetry and compare it with j^μ from above.