Exercises to Relativistic Quantum Field Theory — Sheet 3

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Aufgabe 3.1 *Green's function of Schrödinger's equation* (2 Punkte)

The retarded and advanced *Green's functions* $G_{\mathrm{ret/adv}}$ of the Schrödinger equation are defined by

$$\theta(t-t')\,\psi(t,\vec{x}) = +\mathrm{i}\int \mathrm{d}^3x'\, G_{\mathrm{ret}}(t,\vec{x};t',\vec{x}\,')\,\psi(t',\vec{x}\,'),$$
$$\theta(t'-t)\,\psi(t,\vec{x}) = -\mathrm{i}\int \mathrm{d}^3x'\, G_{\mathrm{adv}}(t,\vec{x};t',\vec{x}\,')\,\psi(t',\vec{x}\,'),$$

i.e. $G_{\mathrm{ret/adv}}$ can be used to transform a wave function given at any time t' to some time $t > t'$ rsp. $t < t'$.

a) Show that $G_{\mathrm{ret/adv}}$ are solutions of the differential equation

$$\left(\mathrm{i}\frac{\partial}{\partial t} + \frac{1}{2m}\Delta - V(\vec{x})\right) G(t,\vec{x};t',\vec{x}\,') = \delta(t-t')\delta^{(3)}(\vec{x}-\vec{x}\,'). \qquad (1)$$

b) The condition (1) can be used as a definition of a Green's function $G(t,\vec{x};t',\vec{x}\,')$ for more general boundary conditions. What are the boundary conditions for $\lim\limits_{\delta t\to 0^+} G_{\mathrm{ret/adv}}(t\pm\delta t,\vec{x};t,\vec{x}\,')$ that fix the retarded rsp. advanced Green's function? What can you say about the behaviour of $G_{\mathrm{ret/adv}}(t,\vec{x};t',\vec{x}\,')$ for $t < t'$ rsp. $t > t'$?

c) Show that every solution of $G(t,\vec{x};t',\vec{x}\,')$ of the integral equation

$$G(t,\vec{x};t',\vec{x}\,') = G_0(t,\vec{x};t',\vec{x}\,') + \int \mathrm{d}\tilde{t}\,\mathrm{d}^3\tilde{x}\; G_0(t,\vec{x};\tilde{t},\tilde{\vec{x}})V(\tilde{\vec{x}})G(\tilde{t},\tilde{\vec{x}};t',\vec{x}\,'), \qquad (2)$$

where $G_0(t,\vec{x};t',\vec{x}\,')$ is a Green's function of the free Schrödinger equation with $V(\vec{x}) = 0$, is a Green's function as defined in (1).

d) Complete the proof of equivalence between the integral equation (2) and the differential equation (1) by showing that every Green's function is a solution of the integral equation if G_0 and G satisfy appropriate boundary conditions such as the ones of the retarded rsp. advanced Green's functions.

Aufgabe 3.2 *Hamiltonian of the classical, free Klein-Gordon field* (1.5 Punkte)

Consider the Lagrangian $\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi)^* - m^2\phi\phi^*$ of the free, complex Klein-Gordon field ϕ.

a) Derive the corresponding Hamiltonian $\mathcal{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}$, with $\pi = \dfrac{\partial\mathcal{L}}{\partial\dot{\phi}}$ denoting the canonical conjugate field to ϕ.

Bitte wenden!

b) Using the plane-wave solution $\phi = \int \mathrm{d}\tilde{p}\Big(a(\vec{p})\mathrm{e}^{-\mathrm{i}px} + b(\vec{p})^* \mathrm{e}^{+\mathrm{i}px}\Big)$ with some (square-integrable) arbitrary complex functions $a(\vec{p})$, $b(\vec{p})$, show that

$$H = \int \mathrm{d}^3x\, \mathcal{H} = \int \mathrm{d}\tilde{p}\, p_0 \Big(|a(\vec{p})|^2 + |b(\vec{p})|^2\Big),$$

where the momentum-space integral is defined by

$$\int \mathrm{d}\tilde{p} \equiv \int \frac{\mathrm{d}^4p}{(2\pi)^4}(2\pi)\delta(p^2 - m^2)\theta(p_0) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p_0}\bigg|_{p_0 = \sqrt{\vec{p}^2 + m^2}}.$$

c) Employing the (time-independent) scalar product

$$(\phi, \chi) \equiv \mathrm{i} \int \mathrm{d}^3x \Big(\phi(x)^*(\partial_0 \chi(x)) - \chi(x)(\partial_0 \phi(x)^*)\Big)$$

of two free Klein-Gordon fields ϕ, χ, is it possible to interpret H as the expectation value of the energy operator $P_0 = \mathrm{i}\partial_0$ of the field modes, i.e. as $(\phi, P_0\phi)$?

Aufgabe 3.3 *Gauge invariance and minimal substitution* (1 Punkt)

The electric and magnetic fields $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi$ and $\vec{B} = \nabla \times \vec{A}$ are invariant under the *gauge transformation* $\phi \to \phi + \frac{\partial \omega}{\partial t}$, $\vec{A} \to \vec{A} - \nabla\omega$ of the scalar and vector potentials, where $\omega(t, \vec{x})$ is an arbitrary function of space and time.

a) Show that Schrödinger's equation of a spinless particle with charge q in the electromagnetic field,

$$\left(\mathrm{i}\frac{\partial}{\partial t} + \frac{1}{2m}\Big(\nabla - \mathrm{i}q\vec{A}(t, \vec{x})\Big)^2 - q\phi(t, \vec{x})\right)\psi(t, \vec{x}) = 0,$$

is invariant under gauge transformations if the following transformation of the wave function ψ is applied simultaneously:

$$\psi(t, \vec{x}) \to \mathrm{e}^{-\mathrm{i}q\omega(t, \vec{x})}\psi(t, \vec{x}).$$

Comment: The introduction of the interaction with the electromagnetic field by the replacements $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \mathrm{i}q\phi$ and $\nabla \to \nabla - \mathrm{i}q\vec{A}$ is called *minimal substitution.*

b) Carrying out the minimal substitution in the free Klein-Gordon equation of a scalar field with charge q we obtain the Klein-Gordon equation with the electromagnetic interaction:

$$\Big((\partial_\mu + \mathrm{i}qA_\mu)(\partial^\mu + \mathrm{i}qA^\mu) + m^2\Big)\Phi(x) = 0$$

with $(A^\mu) = (\phi, \vec{A})$. Show, in analogy to Schrödinger's equation, that this equation is invariant under gauge transformations if the following transformation of the scalar field is applied:

$$\Phi'(x) \to \mathrm{e}^{-\mathrm{i}q\omega(x)}\Phi(x).$$