Group Theory for Physicists

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"The universe is an enormous direct product of representations of symmetry groups."

Hermann Weyl

Contents

Chapter 1

Basic concepts and group theory in quantum mechanics

1.1 Symmetry transformations in quantum mechanics

Classification of symmetry transformations:

- "Space–time symmetries":
  Changes of position or orientation of the observer by translations, reflections, rotations, changing the state of motion, leaving the laws of physics invariant.

- "Internal symmetries":
  Other changes in the qm. states (e.g. interchanging states or particles), leading to physically equivalent systems.

Actions on states and observables

by symmetry operator U on states in Hilbert space $\mathcal{H}$:

$$\begin{aligned}
\text{states } |\psi\rangle \in \mathcal{H} \quad &\xrightarrow{\;U\;} \quad |\psi'\rangle = U|\psi\rangle \in \mathcal{H}, \\
\text{expectation value } \langle A\rangle_\psi = \langle\psi|A|\psi\rangle \quad &\xrightarrow{\;U\;} \quad \langle A'\rangle_{\psi'} = \langle\psi|U^\dagger A' U|\psi\rangle \stackrel{!}{=} \langle A\rangle_\psi, \quad \forall|\psi\rangle \in \mathcal{H}, \\
\text{observable (=operator) } A \quad &\xrightarrow{\;U\;} \quad A' = (U^\dagger)^{-1} A U^{-1}, \\
&\qquad\quad \text{i.e. } A' = UAU^\dagger \text{ if } U = \text{ unitary}, \\
p_{\phi\psi} = |\langle\phi|\psi\rangle|^2 \quad &\xrightarrow{\;U\;} \quad p'_{\phi'\psi'} = |\langle\phi'|\psi'\rangle|^2 \stackrel{!}{=} p_{\phi\psi}.
\end{aligned}$$

= probability to find $|\phi\rangle$ in $|\psi\rangle$ in a measurement ($\|\psi\| = \|\phi\| = 1$)

$\Rightarrow$ U obeys

$$|\langle\phi|\psi\rangle| \;=\; |\langle\phi|U^\dagger U|\psi\rangle| \qquad \forall|\phi\rangle, |\psi\rangle \in \mathcal{H}, \quad \|\psi\| = \|\phi\| = 1. \tag{1.1}$$

Wigner's theorem (non-trivial!)

A symmetry operator U is *unitary* or *antiunitary*,
i.e. $U^\dagger U = \mathbb{1}$ and $U =$ *linear* or *antilinear*.

Examples:

- $U =$ unitary: spatial translation T, rotation R, time evolution $U(t_1, t_0)$, space inversion $\mathcal{P}$, etc.
- $U =$ antiunitary: time reversal $\mathcal{T}$.

Properties of unitary symmetries:

- Symmetry trafos U form a math. "group" G.

  $\hookrightarrow$ Groups are "discrete" ($\mathcal{P}$, etc.) or "continuous" ("Lie groups", e.g. T, R, etc.).

- Operator trafo: $A \to A' = UAU^\dagger$ = similiarity trafo, leaving eigenvalues of A invariant.

  Symmetry: $A' = UAU^\dagger \overset{!}{=} A, \quad U^{-1} = U^\dagger,$
  i.e. $UA = AU, \; [A, U] = 0.$

  $\Rightarrow$ If $|a\rangle$ = eigenstate of A with eigenvalue a: $A|a\rangle = a|a\rangle$,
  then *all* $U|a\rangle$ with $U \in G$ as well:

$$A\big(U|a\rangle\big) \;=\; UA|a\rangle \;=\; a\big(U|a\rangle\big). \tag{1.2}$$

  $\Rightarrow$ Action of sym. ops. characterise eigenvalue spectra of observables, in particular degeneracies.

- Lie group G: $U = U(\theta_1, \ldots, \theta_n)$ = differentiable function of $n \equiv \dim G$ real "group parameters" θ_a.

  Infinitesimal parameters: $(U(0, \ldots, 0) = \mathbb{1}$ by convention)

$$U(\delta\theta_1, \ldots, \delta\theta_n) = \mathbb{1} - \mathrm{i}\delta\theta_a X^a + \mathcal{O}(\delta\theta_a^2), \tag{1.3}$$

$$U(\delta\theta_1, \ldots, \delta\theta_n)^\dagger = \mathbb{1} + \mathrm{i}\delta\theta_a (X^a)^\dagger + \ldots, \tag{1.4}$$

$$\overset{!}{=} U(\delta\theta_1, \ldots, \delta\theta_n)^{-1} = \mathbb{1} + \mathrm{i}\delta\theta_a X^a + \ldots, \quad \text{unitarity!} \tag{1.5}$$

$$\Rightarrow X^a \;=\; (X^a)^\dagger, \quad a = 1, \ldots, n. \tag{1.6}$$

  $\hookrightarrow$ n hermitian operators, i.e. observables characterising the symmetry!

  Summation convention: $\delta\theta_a X^a \equiv \sum_a \delta\theta_a X^a$, i.e. summation over repeatedly appearing indices in products is implicitly assumed.

1.2 Group-theoretical definitions

Definition:

A "group" G is defined by a set of elements $\{g_1, \ldots, g_n\}$ with a mapping $\circ : G \times G \mapsto G$ ("group multiplication") obeying:

(i) $g_1 \circ (g_2 \circ g_3) \;=\; (g_1 \circ g_2) \circ g_3$ (associativity),

(ii) $\exists\, e \in G$ with $g \circ e = g \quad \forall g \in G$ (unit element),

(iii) $\forall g \in G \quad \exists\, g^{-1} \in G$ with $g \circ g^{-1} = e$ (inverse element).

Consequences:

- $g_1 \circ g = g_2 \circ g \quad \Rightarrow \quad g_1 = g_2$ (cancellation law),
- $g \in G$: $\quad e \circ g = g, \quad g^{-1} \circ g = e, \quad \big(g^{-1}\big)^{-1} = g.$

Further notions:

- G is "abelian" if $g_1 \circ g_2 = g_2 \circ g_1 \quad \forall g_1, g_2 \in G$.
- A "group homomorphism" is a mapping $f : G \mapsto G'$ from a group G to a group G' that respects the group multiplication law, i.e.

$$f(\underbrace{g_1 \circ g_2}_{\in G}) \;=\; \underbrace{f(g_1)}_{\in G'} \circ \underbrace{f(g_2)}_{\in G'} \quad \forall g_1, g_2 \in G. \tag{1.7}$$

  The set $\ker(f) = \{g \in G \,|\, f(g) = e' = \text{unit element of } G'\}$ is called "kernel" of f.

- A bijective (injective and surjective) group homomorphism is called "isomorphism". Two groups G, G' connected by an isomorphism are called "isomorphic" ($G \simeq G'$).
- The "direct product group" $G \times G'$ of two group G, G' is the set of all (g, g'), $g \in G$, $g' \in G'$ with the multiplication

$$(g_1, g_1') \circ (g_2, g_2') = (g_1 \circ g_2, g_1' \circ g_2'). \tag{1.8}$$

- A group is called "discrete" if its (#elements) $\equiv |G| \equiv \mathrm{ord}(G) \equiv$ "order of G" is finite or countably infinite.
  $\hookrightarrow$ Elements can be enumerated: $g_1 \equiv e, g_2, g_3, \ldots$
- In a "Lie group" G all elements $U(\theta_1, \ldots, \theta_n)$ are differentiable functions of n real "group parameters" θ_a, $n = \dim G =$ dimension of G.

Examples:

- "Symmetric groups" S_n of all permutations of $(12 \cdots n)$
  = group of order $n!$ which is non-abelian if $n > 2$.

  Elements $P \in S_n$: $\quad P \equiv \begin{pmatrix} 1\,2 & \cdots & n \\ \pi_1\pi_2 & \cdots & \pi_n \end{pmatrix}$ maps $(12 \cdots n) \rightarrow (\pi_1\pi_2 \cdots \pi_n)$.

  $\hookrightarrow$ All P's can be written as products of "transpositions" P_{ij}
  where $\pi_i = j, \pi_j = i$ and $\pi_k = k$ for $k \neq i, j$.

  $\text{sgn}(P) \equiv (-1)^p$ = "signature of P" = $+1$ ("even") or -1 ("odd").

  $\hookrightarrow$ p = (# transpositions) mod 2 needed to achieve P

  "Cayley's theorem": Every finite group is isomorphic to a subgroup of S_n.

- "Alternating group" A_n = subgroup of S_n (order $n!/2$) of all even permutations.

- "Cyclic group" C_n = abelian group of order n generated by one element g:
  $C_n = \{e \equiv g^0 \equiv g^n, g^1, g^2, \ldots, g^{n-1}\}$.

  C_n realised, e.g., by rotations with angles $k \cdot \frac{2\pi}{n}$, $k = 0, 1, \ldots, n-1$, about a fixed axis.

  C_∞ realised by translations with vectors $n \cdot \vec{a}$, $n \in \mathbb{Z}$, with $\vec{a}$ = fixed.

- $\text{GL}(N, \mathbb{K})$ = "general linear group" over $\mathbb{K} = \mathbb{R}, \mathbb{C}$
  = group of invertible $N \times N$ matrices $\in \mathbb{K}^2$.

  $\hookrightarrow$ Non-abelian Lie group of dimension $N^2(\mathbb{R})$ or $2N^2(\mathbb{C})$ for $N > 1$.

1.3 Substructures of groups

1.3.1 Classes

Definition:

Two elements $a, b \in G$ of a group G are called "equivalent" ($a \sim b$) if $\exists g \in G$ with $b = gag^{-1}$. The sets $\mathrm{Cl}(a) = \{b \in G \,|\, b = gag^{-1}\}$ are called "(equivalence) classes" for the "(representative) element" $a \in G$.

Some properties:

- "Equivalence" of group elements as in any set of elements:
  - "reflexivity": $a \sim a$,
  - "symmetry": $a \sim b \quad \Rightarrow \quad b \sim a$,
  - "transitivity": $a \sim b \wedge b \sim c \quad \Rightarrow \quad a \sim c$.
- $\mathrm{Cl}(a) = \mathrm{Cl}(b) \quad \Leftrightarrow \quad a \sim b$.
- The classes $\mathcal{C}_i$ form a "partitioning" of G: $\quad G = \bigcup_i \mathcal{C}_i, \quad \mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for $i \neq j$.

  Convention: $\quad \mathcal{C}_1 = \{e\}$ = class formed by unit element alone.
- In an abelian group each element defines its own class.
- Interpretation: Two elements are equivalent if they have essentially the same multiplication properties.

  Example: Group of linear, invertible mappings in $\mathbb{R}^3$.

  Two matrices A, A' are equivalent if they correspond to the same mapping $\mathcal{A}$ described w.r.t. to two different bases $\{\vec{e}_i\}$, $\{\vec{e}_i'\}$ with $\vec{e}_j = \vec{e}_i' S_{ij}$:

$$\begin{aligned} \vec{x} &= \vec{e}_i x_i = \vec{e}_j' x_j', && \text{i.e.} \quad x_i' = S_{ij} x_j \\ \mathcal{A}\vec{x} &= \vec{e}_i (\mathcal{A}\vec{x})_i = \vec{e}_i A_{ij} x_j = \vec{e}_i' (SAS^{-1})_{ij} x_j' = \vec{e}_i' A_{ij}' x_j' && \text{i.e.} \quad A' = SAS^{-1}. \end{aligned} \tag{1.9}$$

  In particular, rotations about the same angle, but any rotation axis are equivalent.

Example:

Group D_4 = symmetry group of a square (edges A, B, C, D), generated by
ρ = rotation about 90°: $\quad A \to B \to C \to D \to A$,
σ = reflection about a symmetry axis: $\quad A \leftrightarrow B, \quad C \leftrightarrow D$.
$\Rightarrow$ 8 elements $\{e, \rho, \rho^2, \rho^3, \sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\}$ with relations $\rho^4 = \sigma^2 = (\rho\sigma)^2 = e$.
$\Rightarrow$ 5 classes: $\quad \mathcal{C}_1 = \{e\}, \quad \mathcal{C}_2 = \{\rho, \rho^3\}, \quad \mathcal{C}_3 = \{\rho^2\}, \quad \mathcal{C}_4 = \{\sigma, \rho^2\sigma\}, \quad \mathcal{C}_5 = \{\rho\sigma, \rho^3\sigma\}$.

Note: D_4 (order 8) is a subgroup (conserving neighbouring objects) of S_4 (order 24):
$e = (ABCD)$, $\rho = (BCDA)$, $\sigma = (BADC)$, $\rho^2 = (CDAB)$, ...

1.3.2 Subgroups, cosets and Lagrange's theorem

Definition:

A subset $H \subseteq G$ of a group G is a "subgroup" if H is a group with the same product $\circ$ as G. The sets $gH = \{g' \,|\, g' = gh,\ h \in H\}$, $g \in G$, are called "(left) cosets" of H.
"Right cosets" Hg are defined analogously.

Some properties:

- $g_1 H = g_2 H \quad \Leftrightarrow \quad g_1^{-1} g_2 \in H$.

  Proof: "$\Rightarrow$" : $\exists h_1, h_2 \in H : g_1 h_1 = g_2 h_2 \quad \Rightarrow \quad g_1^{-1} g_2 = h_1 h_2^{-1} \in H$
  "$\Leftarrow$" : $g_1^{-1} g_2 \in H \quad \Rightarrow \quad g_1^{-1} g_2 H = H \quad \Rightarrow \quad g_1 H = g_2 H$. #

- Only the coset $hH = H$, $h \in H$, is a subgroup, since $e \notin gH$ if $g \notin H$. (If $e \in gH$, then g is the inverse of some $h \in H$ and hence $g \in H$.)

- All cosets have the same number of elements: $|gH| = |H|$.
  Proof: $\forall g_1, g_2 \in H$ we have $gg_1 = gg_2 \quad \Leftrightarrow \quad g_1 = g_2$.
  $\Rightarrow$ The mapping $g\circ : H \mapsto gH$ is injective. #

- Two left (right) cosets are either equal or disjoint.

- Corollary: "Lagrange's theorem"
  The order of any subgroup H of a finite group G divides the order of G.
  The natural number $[G : H] = |G| : |H|$ is called the "index of H in G".

1.3.3 Invariant subgroups and factor group

Definition:

A subgroup N of a group G is called "invariant" (or "normal") if $N = gNg^{-1}\ \forall g \in G$, written as $N \triangleleft G$.

Comments:

- Equivalent definition: A subgroup is normal if the set of its left cosets equals the set of its right cosets.
  Proof: If $aN = Nb$ for some $b \in G$, then $a \in Nb$.
  Since $a \in Na$, $Nb \cap Na \neq \emptyset \ \Rightarrow\ Na = Nb \ \Rightarrow\ aN = Na$.
  Other direction: $aN = Na \ \Rightarrow$ the sets of left and right cosets are equal. #

- A subgroup N is normal if it contains all $g \in G$ being equivalent to some $h \in N$.

Definition:

Given a normal subgroup N of a group G, then the group of all gN is called the "factor group" G/N.

Note: $gN = Ng$ is essential that all gN form a group:

$$(g_1N)(g_2N) = g_1Ng_2N = g_1g_2NN = (g_1g_2)N. \tag{1.10}$$

Some properties:

- For a finite group G the order of a factor group G/N is equal to the index of the normal subgroup N:

$$\mathrm{ord}(G) = \mathrm{ord}(N) \times [G : N] = \mathrm{ord}(N) \times \mathrm{ord}(G/N). \tag{1.11}$$

- The mapping $f : G \mapsto G/N$ defined by $f(g) = gN$ is a group homomorphism with $N = \ker(f)$.

- "First isomorphism theorem":
  The kernel $\ker(f)$ of a group homomorphism $f : G \mapsto G'$ is a normal subgroup, and $f(G) \simeq G/\ker(f)$.

  Proof:

  a) $H = \ker(f)$ is normal subgroup, since $\forall h \in H$ and $\forall g \in G$ we get

  $$f(ghg^{-1}) = f(g)\underbrace{f(h)}_{=e'}f(g^{-1}) = f(g)f(g^{-1}) = f(gg^{-1}) = f(e) = e'.$$

  $\Rightarrow\ gHg^{-1} \subseteq H$.
  $gHg^{-1} = H$ follows, since $\psi_g : H \mapsto gHg^{-1}$ with $\psi_g(h) = ghg^{-1}$ is injective:

  $$gh_1g^{-1} = gh_2g^{-1} \quad \Leftrightarrow \quad h_1 = h_2.$$

  b) To show $f(G) \simeq G/H$, define mapping $F : G/H \mapsto f(G)$ via $F(gH) = f(g)$. Such an F exists, because if $g_1H = g_2H$, $\exists h_1, h_2 \in H$ with
  $g_1h_1 = g_2h_2,\ g_2 = g_1\underbrace{h_1h_2^{-1}}_{\in H} \ \Rightarrow\ f(g_2) = f(g_1h_1h_2^{-1}) = f(g_1)\underbrace{f(h_1h_2^{-1})}_{e'} = f(g_1)$.
  Show that F is an isomorphism:
  Surjectivity: For each $g' \in f(G)$ $\exists g \in G$ with $g' = f(g) = F(gH)$,
  i.e. also some $gH \in G/H$ with $F(gH) = g'$.
  Injectivity: If $g_1' = g_2'$ for $g_1' = F(g_1H)$, $g_2' = F(g_2H)$, we have
  $e' = (g_1')^{-1}g_2' = F(g_1H)^{-1}F(g_2H) = f(g_1)^{-1}f(g_2)$
  $= f(g_1^{-1})f(g_2) = f(g_1^{-1}g_2)$, i.e. $g_1^{-1}g_2 \in H = \ker(f)$.
  $\Rightarrow\ g_1H = g_2H$. #

1.4 Group representations

Motivation:

Abstract symmetry trafo $g \in G \xrightarrow{\text{represented as}}$ operator $U(g)$ acting on states $|\psi\rangle \in \mathcal{H}$.

$\Rightarrow$ Issues:

- Which states $|\psi\rangle$ are symmetry connected,
  i.e. how are the subspaces $U_\psi = \{U(g)|\psi\rangle, g \in G\}$ characterised?
- Which types of U_ψ do exist for given G?
- What are appropriate basis states $|\phi_k\rangle$ making the action of $U(g)$ transparent?

$\hookrightarrow$ Answered by "representation theory of groups"!

Definition:

A "representation D of a group G on a vector space V" is a homomorphism $D : G \mapsto \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ = "general linear group on V" = group of invertible linear mappings on V, with

$$D(g_1 \circ g_2) = D(g_1)\, D(g_2) \quad \forall g_1, g_2 \in D, \tag{1.12}$$

$\Rightarrow$ In particular: $D(e) = \mathbb{1}$ = unit operator and $D(g^{-1}) = D(g)^{-1}$.

Types of representations:

- $\dim D \equiv \dim V < \infty$: $D(g)$ = matrices with the usual matrix multiplication.
- $\dim D = \infty$, but countable: $D(g)$ = infinitely large matrices,

$$D = \begin{pmatrix} D_{11} & D_{12} & \cdots \\ D_{21} & D_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.13}$$

- $\dim D = \infty$, not countable:
  typical of "extended Hilbert spaces $\mathcal{H}$" with improper states.
  Example: functions $\psi(x)$ of $x \in \mathbb{R}$, $T(a)$ = translation by a constant a,

$$T(a)\, \psi(x) = \psi(x - a) = \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} \left(-a \frac{\partial}{\partial x}\right)^n}\, \psi(x). \tag{1.14}$$

$\hookrightarrow$ trafo represented by a differential operator

Further notions:

- D is called "unitary" if $D(g)$ = unitary $\forall g \in G$ and V is a unitary vector space.
- D is called "faithful" if $g_1 \neq g_2$ implies $D(g_1) \neq D(g_2)$.
  $\hookrightarrow$ D carries the full information of G.

  Note: If $D \neq$ faithful, $D(g) = \mathbb{1}$ for some $g \neq e$.
  Extreme case: $D(g) = \mathbb{1}\ \ \forall g \in G$, "trivial representation".
- D_1 and D_2 are "equivalent" ($D_1 \simeq D_2$) if $\exists$ linear mapping S with

$$S\,D_1(g)\,S^{-1} \;=\; D_2(g) \quad \forall g \in G \qquad \text{(common similiarity trafo for all } g!) \tag{1.15}$$

- "Direct sum representation" $D_1 \oplus D_2$ on $V_1 \oplus V_2$ for two representations D_i on V_i:

$$\begin{aligned} D_1{\oplus}D_2(g)\ \big(|\psi_1\rangle, |\psi_2\rangle\big) &= \big(D_1(g)|\psi_1\rangle, D_2(g)|\psi_2\rangle\big), \qquad |\psi_i\rangle \in V_i, \\ \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} &= \begin{pmatrix} D_1(g)|\psi_1\rangle \\ D_2(g)|\psi_2\rangle \end{pmatrix}, \end{aligned} \tag{1.16}$$

  i.e. actions of D_1, D_2 "blockwise independent".
- D is called "reducible" if $\exists$ non-trivial invariant subspace $V_1 \subset V$ ($V_1 \neq V$), i.e.

$$D(g)v_1 \in V_1 \qquad \forall g \in G, \quad v_1 \in V_1. \tag{1.17}$$

  Otherwise D is called "irreducible".

  In detail:

  – D = reducible $\Leftrightarrow$ $\exists$ linear mapping S with

$$D(g) = S \begin{pmatrix} D_1(g) & X(g) \\ 0 & Y(g) \end{pmatrix} S^{-1} \quad \forall g \in G.$$

  S can be determined by a basis change in V so that $\big\{ \underbrace{|\phi_1\rangle, \dots, |\phi_{n_1}\rangle}_{\text{basis of } V_1}, |\phi_{n_1+1}\rangle, , \dots, |\phi_n\rangle \big\} =$ basis of V.

  – D = irreducible $\Leftrightarrow$ $V_\psi = \big[D(g)|\psi\rangle, g \in G\big] = V \quad \forall |\psi\rangle \in V$ with $|\psi\rangle \neq 0$.

  The symmetry-connected vectors $D(g)|\psi\rangle$ of any $|\psi\rangle \neq 0$ span the full representation space V, i.e. symmetry trafos transform all basis vectors $|\phi_k\rangle$ of V non-trivially into each other.

  Basis of $V = \{|\phi_1\rangle, \dots, |\phi_n\rangle\}$ = "symmetry multiplet".

- Finite-dimensional unitary representations are "fully reducible", i.e. $\exists S$ with

$$D(g) = S \begin{pmatrix} D^{(1)}(g) & 0 & \dots & \\ 0 & D^{(2)}(g) & \dots & \\ \vdots & & \ddots & \\ & & & D^{(I)}(g) \end{pmatrix} S^{-1} \qquad \forall g \in G, \quad D^{(i)} = \text{irreducible.} \tag{1.18}$$

Proof:

a) If $D =$ irreducible, there is nothing to prove.

b) $D =$ reducible. $\Rightarrow$ $\exists$ invariant subspace $V_1 \subset V$ $(V_1 \neq V)$.
$D =$ unitary, i.e. $\exists$ scalar product in V.
$\hookrightarrow$ Decompose $V = V_1 \oplus V_1^{\perp}$,

$$|\psi\rangle = \underbrace{|\psi_1\rangle}_{\in V_1} + \underbrace{|\psi_1^{\perp}\rangle}_{\in V_1^{\perp}}, \quad \langle\psi_1|\psi_1^{\perp}\rangle = 0.$$

c) Show that $V_1^{\perp} =$ invariant subspace:

$$\langle\psi_1|D(g)|\psi_1^{\perp}\rangle \;=\; \langle\underbrace{D(g)^{\dagger}\psi_1}_{\in V_1}|\underbrace{\psi_1^{\perp}}_{\in V_1^{\perp}}\rangle \;=\; 0 \quad \forall\, |\psi_1\rangle \in V_1,\; |\psi_1^{\perp}\rangle \in V_1^{\perp}.$$

$\Rightarrow$ $D(g)|\psi_1^{\perp}\rangle \in V_1^{\perp}$.

$$\Rightarrow\; D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \text{ in basis } \{\underbrace{|\phi_1\rangle, \dots, |\phi_{n_1}\rangle}_{\text{basis of } V_1}, \underbrace{|\phi_{n_1+1}\rangle, , \dots, |\phi_n\rangle}_{\text{basis of } V_1^{\perp}}\}.$$

d) Repeat procedure for D_1 and D_2 if D_1 or D_2 is reducible.

#

- "Product representation" $D_1 \otimes D_2$ on $V_1 \otimes V_2$ for two representations D_i on V_i:

$$D_1{\otimes}D_2(g)\;\Big(\underbrace{|\psi_1\rangle \otimes |\psi_2\rangle}_{\in V_1\otimes V_2,\;\dim V_1\otimes V_2 = \dim V_1 \cdot \dim V_2}\Big) \;=\; D_1(g)|\psi_1\rangle \otimes D_2(g)|\psi_2\rangle, \qquad |\psi_i\rangle \in V_i. \tag{1.19}$$

Note: $D_1 \otimes D_2$ in general is reducible even if D_i are irreducible.

But: $D_1 \otimes D_2$ is fully reducible if D_1, D_2 are unitary!

$\Rightarrow$ $\exists$ "Clebsch–Gordan decomposition"

$$D_1 \otimes D_2 \;=\; D^{(1)} \oplus D^{(2)} \oplus \dots \oplus D^{(I)}, \tag{1.20}$$

by decomposing the matrices $D_1 \otimes D_2(g)$ into irreducible building blocks $D^{(i)}(g)$ by an appropriate similiarity trafo:

$$D_1{\otimes}D_2(g) = S \begin{pmatrix} D^{(1)}(g) & 0 & \cdots \\ 0 & D^{(2)}(g) & \cdots \\ \vdots & & \ddots \end{pmatrix} S^{-1}. \tag{1.21}$$

Definition: "Group characters"

The "character" $\chi_D(g)$ of a representation matrix $D(g)$ of a representation of an element g of a group G is defined by the trace of $D(g)$:

$$\chi_D(g) = \mathrm{tr}\{D(g)\} = \sum_{i=1}^{\dim D} D_{ii}(g). \tag{1.22}$$

Some properties:

- Characters depend on the group G *and* on the representation $D(G)$.
- Characters are functions of classes, i.e. if $g_1, g_2 \in \mathcal{C}_k$ then $\chi_D(g_1) = \chi_D(g_2) \equiv \chi_D(\mathcal{C}_k)$.
  Proof: $\exists g \in G$ with $g_1 = g g_2 g^{-1}$.

$$\begin{aligned} \Rightarrow \quad \chi_D(g_1) &= \mathrm{tr}\{D(g_1)\} = \mathrm{tr}\{D(g g_2 g^{-1})\} = \mathrm{tr}\{D(g)D(g_2)D(g^{-1})\} \\ &= \mathrm{tr}\{D(g^{-1})D(g)D(g_2)\} = \mathrm{tr}\{D(g)^{-1}D(g)D(g_2)\} = \mathrm{tr}\{D(g_2)\} \\ &= \chi_D(g_2). \qquad \# \end{aligned}$$

- Special case unit element: $\chi_D(\mathcal{C}_1) = \mathrm{tr}\{D(e)\} = \mathrm{tr}\{\mathbb{1}\} = \dim D$.
- Note: Characters in general do *not* form representations,
  since in general $\mathrm{tr}\{AB\} \neq \mathrm{tr}\{A\} \cdot \mathrm{tr}\{B\}$.

  But: Determinants of $D(g)$ form another (one-dimensional) representation:

$$\det\{D(g_1)D(g_2)\} = \det\{D(g_1)\} \cdot \det\{D(g_2)\}. \tag{1.23}$$

- Characters of outer product matrices are products of characters of individual factors:

$$\begin{aligned} \chi_{D_1 \otimes D_2}(g) &= \mathrm{tr}\{(D_1 \otimes D_2)(g)\} = \sum_{a=1}^{\dim D_1 \otimes D_2} (D_1 \otimes D_2)_{aa}(g) \\ &= \sum_{i=1}^{\dim D_1} \sum_{j=1}^{\dim D_2} D_{1,ii}(g) D_{2,jj}(g) = \left(\sum_{i=1}^{\dim D_1} D_{1,ii}(g) \right) \left(\sum_{j=1}^{\dim D_2} D_{2,jj}(g) \right) \\ &= \chi_{D_1}(g) \cdot \chi_{D_2}(g). \end{aligned} \tag{1.24}$$

1.5 Implications for quantum-mechanical systems

Consider qm. system with Hamiltonian $\hat{H}$ with the symmetry group G:

$$[\hat{H}, U(g)] \;=\; 0, \qquad g \in G, \quad U(g) = \text{symmetry operator on } \mathcal{H}, \tag{1.25}$$
$$= \text{unitary (antiunitarity only for time reversal)}.$$

$\Rightarrow$ $U = \{U(g) \mid g \in G\}$ forms a unitary representation of G on $\mathcal{H}$.

$\Rightarrow$ U is fully reducible, i.e. can be brought to block-diagonal form by an appropriate choice of basis in $\mathcal{H}$:

$$U(g) = \begin{pmatrix} U^{(1)}(g) & 0 & \cdots \\ 0 & U^{(2)}(g) & \cdots \\ \vdots & & \ddots \end{pmatrix}, \qquad U^{(r)} = \text{irreducible representation of } G \tag{1.26}$$
(which can be the same for various r values),

$$\dim U^{(r)} = n_r. \tag{1.27}$$

Consider an arbitray energy eigenstate $|E, a\rangle$, $a = 1, \ldots, n_E$,
n_E = degree of degeneracy of E.

$\Rightarrow$ All $U(g)|E, a\rangle$ are energy eigenstates to energy E:

$$\hat{H}\left(U(g)|E, a\rangle\right) \;=\; U(g)\hat{H}|E, a\rangle \;=\; E\left(U(g)|E, a\rangle\right), \qquad a = 1, \ldots, n_E. \tag{1.28}$$

$\Rightarrow$ $U(g)|E, a\rangle$ is linear combination of $|E, b\rangle$, $b = 1, \ldots, n_E$:

$$U(g)|E, a\rangle \;=\; \sum_{b=1}^{n_E} |E, b\rangle\, D_{ba}(g), \qquad \text{normalisation: } \langle E, a|E, b\rangle = \delta_{ab}. \tag{1.29}$$

$\Rightarrow$ $D = \{D(g) \mid g \in G\} = n_E$-dim. unitary representation of G
on the "degeneracy space" spanned by $\{|E, a\rangle\}_{a=1}^{n_E}$.

$\Rightarrow$ 2 possible cases:

a) D is one of the irreducible representations $U^{(r)}$ of U.

   $\Rightarrow$ Degeneracy of states $|E, a\rangle$ is a consequence of the sym. group G of the system.

b) D is some direct-sum representation $U^{(r_1)} \oplus U^{(r_2)} \oplus \cdots \oplus U^{(r_E)}$ with dimension $n_E = n_{r_1} + n_{r_2} + \cdots + n_{r_E}$.

   $\Rightarrow$ Degeneracy between basis states (multiplets) of different $U^{(r_i)}$ blocks is "accidental", i.e. not implied by group G.

   Note: Most likely G does not exhaust the full symmetry of the system.
   $\hookrightarrow$ Find larger symmetry group until no accidental symmetries remain.

$\Rightarrow$ Block form of $\hat{H}$:

$$\hat{H} \;=\; \begin{pmatrix} E_1 \cdot \mathbb{1}_{n_1} & 0 & \cdots \\ 0 & E_2 \cdot \mathbb{1}_{n_2} & \cdots \\ \vdots & & \ddots \end{pmatrix}, \tag{1.30}$$

with $E_r = E_{r'}$ $(r \neq r')$ only for accidental symmetries.

Reduction of symmetries

Typical case: $$\underbrace{\hat{H}'}_{\text{new Hamiltonian}} = \underbrace{\hat{H}}_{\text{as above}} + \underbrace{\delta\hat{H}}_{\substack{\text{new contribution,}\\ \text{e.g., by switching on elmg. fields}}}$$

Suppose $\delta\hat{H}$ does not respect the full symmetry group G.
$\hookrightarrow$ $\hat{H}'$ has symmetry group $G' \subset G$ $(G' \neq G)$.

$\Rightarrow$ Relation between irreducible representations of G' and G?

- Representations of G automatically deliver representations of G':
  $U(G) \to U(G')$ by subset of trafos.
- But: $U(G')$ in general is reducible, even if $U(G)$ is irreducible.
  Multiplet of U:

  $g' \in G'$ only mix subsets of $|\psi_k\rangle$ in a non-trivial way.

  $$\begin{pmatrix} |\psi_1\rangle \\ \vdots \\ |\psi_{n'}\rangle \\ |\psi_{n'+1}\rangle \\ \vdots \\ |\psi_{n'}\rangle \end{pmatrix}$$

  $g \in G$ mix all $|\psi_k\rangle$ in a non-trivial way.

  Less states $|\psi_k\rangle$ are symmetry connected, i.e. degrees of degeneracy between energy eigenstates can be reduced.

Example: 2-dim. qm. harmonic oscillator

Hamiltonian for particle of mass m:

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{m}{2}\left(\omega_1^2\hat{x}^2 + \omega_2^2\hat{y}^2\right) = \sum_{k=1,2} \hbar\omega_k \left(a_k^\dagger a_k + \tfrac{1}{2}\right). \tag{1.31}$$

Energy eigensystem:

$$|n_1, n_2\rangle = |n_1\rangle|n_2\rangle, \qquad |n_k\rangle = \left(a_k^\dagger\right)^{n_k}|0\rangle, \qquad n_1, n_2 \in \mathbb{N}_0, \tag{1.32}$$

$$\hat{H}|n_1, n_2\rangle = E_{n_1,n_2}|n_1, n_2\rangle, \qquad E_{n_1,n_2} = \hbar\omega_1\left(n_1 + \tfrac{1}{2}\right) + \hbar\omega_2\left(n_2 + \tfrac{1}{2}\right). \tag{1.33}$$

Symmetry and degeneracy:

- Symmetric case, $\omega_1 = \omega_2 \equiv \omega$:
  $E_{n_1,n_2} = E_n = \hbar\omega(n+1)$ with $n = n_1+n_2$ is $(n+1)$-fold degenerate due to symmetry:

$$\hat{U} : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to U\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad [\hat{H}, \hat{U}] = 0, \tag{1.34}$$

$$U(\phi_0, \phi_1, \phi_2, \phi_3) = \mathrm{e}^{-\mathrm{i}\phi_0}\,\exp\{-\mathrm{i}\phi_k\sigma_k\} = \text{unitary } 2\times 2 \text{ matrix}, \quad \phi_k \in [0, 2\pi).$$

  $\hat{U}$ comprises:

  - rotations about $\vec{e}_z$ axis: $\exp\{-\mathrm{i}\phi_2\sigma_2\}$,
  - reflections $x \to -x$, $y \to -y$,
  - phase transformations of a_k: $a_k \to \mathrm{e}^{\mathrm{i}(\phi_3 \pm \phi_0)}\, a_k$,
  - complex transformations mixing coordinates and momenta.

  Classification of states $|n_1, n_2\rangle$ by a maximal set of commuting symmetry operators:
  E.g. take rotations about $\vec{e}_z$ axis.
  $\hookrightarrow$ Basis change $\{|n_1, n_2\rangle\} \to \{|n; m\rangle'\}$ to eigenstates of $\hat{H}$ and $\hat{L}_3$:

$$\hat{H}\,|n; m\rangle' = E_n\,|n; m\rangle', \qquad \hat{L}_3\,|n; m\rangle' = \hbar m\,|n; m\rangle'. \tag{1.35}$$

- Unsymmetric case, $\omega_1 \neq \omega_2$:
  Symmetry reduced to two independent (commuting) phase transformations:

$$a_k \to \mathrm{e}^{-\mathrm{i}\phi_k}\, a_k, \quad \phi_k \in [0, 2\pi). \tag{1.36}$$

  $\hookrightarrow$ Only “accidental” degeneracy for $\frac{\omega_1}{\omega_2}$ = rational.

1.6 Schur's lemmas

$\hookrightarrow$ Mathematical statements on irreducible representations $D(G)$ on V:

(i) If there is a linear mapping $S : V \mapsto V$ with $D(g)\,S = S\,D(g)$, i.e. $[D(g), S] = 0$, $\forall g \in G$, and if D is irreducible, then $S = \lambda \cdot \mathbb{1}$.

(ii) If there is a linear mapping $S : V_1 \mapsto V_2$ with $D_1(g)\,S = S\,D_2(g)$ $\forall g \in G$ and if D_1, D_2 are irreducible, then either $S = 0$ or $S =$ invertible (i.e. $D_1 \simeq D_2$).

Note: Schur's lemmas hold for vector spaces with $\dim < \infty$, and also for $\dim = \infty$ if the representations are unitary.

Proof:

(i) $\exists$ eigenvalue $\lambda \in \mathbb{C}$ with eigenvector $|\psi\rangle \neq 0$: $\quad S|\psi\rangle = \lambda|\psi\rangle$.

(This step requires the unitarity of D for $\dim V = \infty$.)

$\Rightarrow\ (S - \lambda \cdot \mathbb{1})\,D(g)|\psi\rangle = D(g)\,\underbrace{(S - \lambda \cdot \mathbb{1})|\psi\rangle}_{=0} = 0 \quad \forall g \in G.$

$\Rightarrow\ D(g)|\psi\rangle$ are all eigenstates of S with eigenvalue λ.
But the eigenspace of $\lambda \equiv V_\lambda \overset{!}{=} V$, since $D =$ irreducible.

$\Rightarrow\ S = \lambda \cdot \mathbb{1}$.

(ii) $K_1 \equiv \{\,|\phi\rangle \in V_1 \mid S|\phi\rangle = 0\,\} =$ kernel of S

is invariant under D_1: $\forall|\phi\rangle \in K_1 :\ S\,D_1(g)|\phi\rangle = D_2(g)\,S|\phi\rangle = 0 \Rightarrow D_1(g)|\phi\rangle \in K_1$.

$W_2 \equiv \{\,|\psi\rangle \in V_2 \mid |\psi\rangle = S|\phi\rangle, |\phi\rangle \in V_1\,\} =$ range of S

is invariant under D_2: $\quad \forall|\psi\rangle \in W_2 :\quad D_2(g)|\psi\rangle = D_2(g)\,S|\phi\rangle = S\,D_1(g)|\phi\rangle \in W_2$.

D_1, $D_2 =$ irreducible. $\quad \Rightarrow\ K_1 = V_1$ or $\{0\}$, $\quad W_2 = V_2$ or $\{0\}$.

a) $K_1 = V_1$. $\quad \Rightarrow\ W_2 = 0$, i.e. $S = 0$.

b) $K_1 = \{0\}$. $\quad \Rightarrow\ S =$ invertible, i.e. $W_2 \neq \{0\}$. $\quad \Rightarrow\ W_2 = V_2$,
i.e. $\dim V_1 = \dim V_2$, $\quad S\,D_1(g)\,S^{-1} = D_2(g)\ \ \forall g \in G$.

#

"Inverse statement" to (i):

Let $D(G)$ be a unitary representation of the group G. If $[D(g), S] = 0 \; \forall g \in G$ implies that $S = \lambda \cdot \mathbb{1}$, then D is irreducible.

Proof: (indirect!)

If D = reducible, then D = fully reducible (since unitary) and $\exists$ basis of V so that

$$D(g) = \begin{pmatrix} D^{(1)}(g) & 0 & \cdots & \cdots \\ 0 & D^{(2)}(g) & \cdots & \cdots \\ \vdots & \vdots & \ddots & \\ \vdots & \vdots & & D^{(I)}(g) \end{pmatrix} \quad \forall g \in G.$$

$$\Rightarrow \; S = \begin{pmatrix} \lambda_1 \cdot \mathbb{1}_{n_1} & 0 & \cdots \\ 0 & \lambda_2 \cdot \mathbb{1}_{n_2} & \\ \vdots & & \ddots \end{pmatrix}, \quad \lambda_1 \neq \lambda_2, \quad \text{obeys } [D(g), S] = 0.$$

#

Consequences for abelian groups:

All irreducible representations of abelians groups are 1-dimensional.

Proof:

$[D(g), D(g')] = 0 \;\; \forall g, g' \in G$ (= abelian).

$\Rightarrow$ All $D(g) = \underbrace{d(g)}_{\in \mathbb{C}} \cdot \mathbb{1}$ if D = irreducible (Schur's lemma).

But $D(g) = \begin{pmatrix} d(g) & 0 & \cdots \\ 0 & d(g) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$ = irreducible only if $\dim D = 1$.

#

Example: 3-dim. representation of S_3 (=non-abelian group of lowest order)

6 permutations of 3 objects ABC: $g_{123} = e,\ g_{231},\ g_{312},\ g_{132},\ g_{321},\ g_{213}$.

Unitary representation via permutation matrices:

$$D(e) = \mathbb{1}_3, \quad D(g_{231}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{etc.} \tag{1.37}$$

Obviously an invariant subspace $[\vec{n}_1]$ is spanned by $\vec{n}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,

i.e. D is reducible.

$\hookrightarrow$ Choose new basis of $V = \mathbb{R}^3$: $\vec{n}_1, \quad \vec{n}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{n}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

$$S\,D(g)\,S^{-1} = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & D'(g) \end{pmatrix}, \qquad S = (\vec{n}_1, \vec{n}_2, \vec{n}_3) = \text{unitary}. \tag{1.38}$$

This defines a new 2-dim. representation D':

$$\begin{aligned} D'(e) &= \mathbb{1}_2, & D'(g_{231}/g_{312}) &= \begin{pmatrix} -\frac{1}{2} & \mp\frac{\sqrt{3}}{2} \\ \pm\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \\ D'(g_{132}/g_{321}) &= \begin{pmatrix} +\frac{1}{2} & \pm\frac{\sqrt{3}}{2} \\ \pm\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, & D'(g_{213}) &= \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}. \end{aligned} \tag{1.39}$$

Check (ir)reducibility of D' via inverse of Schur's lemma:

Ansatz: $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$.

$$\left.\begin{aligned} [T, D'(g_{213})] &\stackrel{!}{=} 0 \quad \Rightarrow\ t_{12} = t_{21} = 0. \\ [T, D'(g_{231})] &\stackrel{!}{=} 0 \quad \Rightarrow\ t_{11} = t_{22}. \end{aligned}\right\} \Rightarrow\ T \propto \mathbb{1}_2. \tag{1.40}$$

$\Rightarrow$ $D' =$ irreducible.

1.7 Real, pseudoreal, and complex representations

Let $D(G)$ be a representation of some group G.
$\Rightarrow$ The set $D(G)^*$ of complex conjugate matrices forms another representation.
$\hookrightarrow$ Question: Is $D(G)^*$ equivalent to $D(G)$ or not?

Definition:

Let $D(G)$ be a *unitary, irreducible* representation of the group G.

(i) $D(G)$ is "complex" if $D(G)$ and $D(G)^*$ are *not* equivalent.

(ii) $D(G)$ is "real" or "pseudoreal" if $D(G)$ and $D(G)^*$ are equivalent:

$$\exists S \ \text{ with } \ D(g)^* = S\, D(g)\, S^{-1} \quad \forall\, g \in G. \tag{1.41}$$

$D(G)$ is real/pseudoreal if $S^{\mathrm{T}} = \pm S$.

Some important properties:

a) $D(G)$ is complex. $\Leftrightarrow$ Not all characters are real.
$\hookrightarrow$ This obviously identifies complex representations, and $\chi_{D^*}(g) = \chi_D^*(g)$.

b) If (1.41) holds, then $S^{\mathrm{T}} = \pm S$.

Proof:

Use unitarity of $D(g)$, so that $D(g)^* = D(g^{-1})^{\mathrm{T}}$:

$$\begin{aligned} D(g) &= D(g^{-1})^\dagger = \left(D(g^{-1})^*\right)^{\mathrm{T}} = \left(S\, D(g^{-1})\, S^{-1}\right)^{\mathrm{T}}, && (1.41) \text{ for } g^{-1} \\ &= (S^{-1})^{\mathrm{T}}\, D(g^{-1})^{\mathrm{T}}\, S^{\mathrm{T}} = (S^{-1})^{\mathrm{T}}\, D(g)^*\, S^{\mathrm{T}}, && \text{unitarity of } D(g) \\ &= (S^{-1})^{\mathrm{T}}\, S\, D(g)\, S^{-1}\, S^{\mathrm{T}} \\ &= \left(S^{-1}\, S^{\mathrm{T}}\right)^{-1}\, D(g)\, S^{-1}\, S^{\mathrm{T}} = M^{-1}\, D(g)\, M, && M \equiv S^{-1}\, S^{\mathrm{T}}. \end{aligned}$$

$\Rightarrow$ $[M, D(g)] = 0 \ \ \forall g \in G$ and thus $M = \lambda \cdot \mathbb{1}$ according to Schur's lemma.

$$\Rightarrow \ S^{\mathrm{T}} = \lambda S = \lambda^2 S^{\mathrm{T}}, \qquad \lambda^2 = 1, \qquad \lambda = \pm 1.$$

#

c) If $D(G)$ is real/pseudoreal, S can be chosen unitary.

Proof:

Again based on unitarity of $D(g)$:

$$S = D(g^{-1})^*\, S\, D(g), \quad S^\dagger = D(g)^\dagger\, S^\dagger\, D(g^{-1})^{\mathrm{T}}$$

$$\Rightarrow \quad D(g)\, S^\dagger S = \underbrace{D(g)\, D(g)^\dagger}_{=\mathbb{1}}\, S^\dagger\, \underbrace{D(g^{-1})^{\mathrm{T}}\, D(g^{-1})^*}_{=\mathbb{1}}\, S\, D(g) = S^\dagger S\, D(g).$$

$\Rightarrow$ $[S^\dagger S, D(g)] = 0 \ \ \forall g \in G$ and thus $S^\dagger S = \sigma \cdot \mathbb{1}$ according to Schur's lemma.
$\hookrightarrow$ Redefine $S \to S/\sqrt{\sigma}$, so that $S^\dagger S = \mathbb{1}$ and $S^{-1} \to S^{-1}/\sqrt{\sigma}$,
i.e. (1.41) stays intact.

#

d) If the representation $D(G)$ is real, then all $D(g)$ can be chosen real.

Sketch of proof:

According to b) and c), (1.41) holds with some *symmetric and unitary* S.
$\hookrightarrow$ $\exists$ *symmetric and unitary* matrix T with $S = T^2$ (proof $\to$ linear algebra).
Define new representation $D'(g) = T\,D(g)\,T^{-1}$, so that ($T = T^{\mathrm{T}}$, $T^\dagger = T^*$)

$$\begin{aligned} D'(g)^* &= \left(T\,D(g)\,T^{-1}\right)^* = T^*\,D(g)^*\,T = T^*\,S\,D(g)\,S^{-1}\,T \\ &= \underbrace{T^*\,T}_{=\mathbb{1}}\,\underbrace{T\,D(g)\,T^*}_{=D'(g)}\,\underbrace{T^*\,T}_{=\mathbb{1}} = D'(g), \end{aligned}$$

i.e. $D'(g) = \text{real} \;\; \forall g \in G$. #

e) For real/pseudoreal $D(G)$, there is a bilinear invariant product $(\,.\,,.\,)$:

$$(x, y) \equiv x^{\mathrm{T}}\,S\,y, \quad x, y \in V, \tag{1.42}$$
$$(x, y) = (D(g)x, D(g)y) \quad \forall g \in G. \tag{1.43}$$

Proof:

Use unitarity of $D(g)$, so that $D(g)^{\mathrm{T}} = D(g^{-1})^*$:

$$\begin{aligned} (D(g)x, D(g)y) &= x^{\mathrm{T}}\,D(g)^{\mathrm{T}}\,S\,D(g)\,y = x^{\mathrm{T}}\,\underbrace{D(g^{-1})^*}_{=\,S\,D(g^{-1})\,S^{-1}}\,S\,D(g)\,y \\ &= x^{\mathrm{T}}\,S\,D(g)^{-1}\,D(g)\,y = x^{\mathrm{T}}\,S\,y = (x, y). \end{aligned}$$ #

Chapter 2

Finite groups

2.1 Multiplication tables

Recall the cancellation law: If $a, b, p \in G$ and $pa = pb$ (or $ap = bp$), the $a = b$. Proof: multiply by p^{-1} from the left (from the right). This implies the "rearrangement lemma":

- If $\{g_1, g_2, \ldots, g_{n_G}\}$ are the elements of a finite group G of order n_G, then $\forall p \in G$, $\{pg_1, pg_2, \ldots, pg_{n_G}\} = \{g_{\sigma_p(1)}, g_{\sigma_p(2)}, \ldots, g_{\sigma_p(n_G)}\}$ is a permutation σ_p of the elements.
- If $a \neq e$, $\sigma_a(k) \neq k \forall k$.
  $\Rightarrow$ the permutation leaves no element invariant.

All possible products of two elements can be written as an $n_G \times n_G$ table:

	$g_1 = e$	$\cdots$	g_j	$\cdots$	g_{n_G}
$g_1 = e$	e	$\cdots$	g_j	$\cdots$	g_{n_G}
$\vdots$	$\vdots$	$\ddots$			$\vdots$
g_i	g_i		$g_i g_j$		$g_i g_{n_G}$
$\vdots$	$\vdots$			$\ddots$	$\vdots$
g_{n_G}	g_{n_G}	$\cdots$	$g_{n_G} g_j$	$\cdots$	$g_{n_G} g_{n_G}$

- The multiplication table characterises the group completely.
- In each row and in each column, every group element appears exactly once, i.e. each row and each column is a permutation of the elements of the group (rearrangement lemma).
  $\Rightarrow$ Cayley's theorem: every finite group of n_G elements is isomorphic to a subgroup of the permutation group S_{n_G}.

Examples:

In the case of groups with 2 rsp. 3 elements, the multiplication tables are unique (we leave out the redundant first row and column):

$C_2 \simeq S_2$:

e	A
A	e

rsp. C_3:

e	A	B
A	B	e
B	e	A

In the case of 4 elements, there are two possibilities:

$C_2 \otimes C_2$:

e	A	B	C
A	e	C	B
B	C	e	A
C	B	A	e

and C_4:

e	A	B	C
A	B	C	e
B	C	e	A
C	e	A	B

Choosing $AA = B$ fixes the table immediately. The case $AA = C$ is redundant, because relabelling B and C shows that this is the same case as $AA = B$. Choosing $AA = e$ still leaves the options $BB = e$ and $BB = A$. But $BB = A$ is equivalent to the case $AA = B$ upon relabelling A and B.

A compact way to characterise a finite group is to define its generating elements, i.e. the elements from which all other elements can be constructed by multiplication.

Examples:

- C_4: All elements are generated by a single element A: $\langle A | A^4 = e \rangle$,
- $C_2 \otimes C_2$: $\langle A, B | A^2 = B^2 = e, AB = BA \rangle$.

This is called a presentation. General form: $\langle \text{generating elements} | \text{relations} \rangle$.

2.2 Unitarity theorem

Theorem: All representations of finite groups are equivalent to unitary representations.
Let $D(g)$ be a representation on a vector space V and define $H \equiv \sum_g D^\dagger(g) D(g)$. Properties:

- $D^\dagger(g') H D(g') = \sum_g D^\dagger(g') D^\dagger(g) D(g) D(g') = \sum_g D^\dagger(gg') D(gg') = H$ (rearrangement lemma),
- H is hermitian, $H = H^\dagger$,
- $\forall$ eigenvectors $|h_i\rangle$, $\langle h_i | h_i \rangle = 1$, with eigenvalue h_i, $i = 1, \dots$:

$$h_i = \langle h_i | H | h_i \rangle = \sum_g \langle h_i | D^\dagger(g) D(g) | h_i \rangle = \sum_g \| D(g) | h_i \rangle \|^2 > 0. \tag{2.1}$$

  $\Rightarrow$ All eigenvalues h_i of H are positive.

- $\exists$ unitary P such that $H = P^\dagger \operatorname{diag}(h_1, \dots) P$
  $\Rightarrow H = S^\dagger S$ with $S = \operatorname{diag}(\sqrt{h_1}, \dots) P$.

The representation $U(g) = SD(g)S^{-1}$ is unitary and $U \simeq D$:

$$\begin{aligned}\langle x|U^\dagger(g)U(g)|y\rangle &= \langle x|(S^{-1})^\dagger D^\dagger(g)\underbrace{S^\dagger S}_{H} D(g)S^{-1}|y\rangle \\ &= \langle x|(S^{-1})^\dagger H S^{-1}|y\rangle \\ &= \langle x|\underbrace{(S^{-1})^\dagger S^\dagger}_{=(SS^{-1})^\dagger=\mathbb{1}} SS^{-1}|y\rangle = \langle x|y\rangle \quad \forall |x\rangle, |y\rangle \in V.\end{aligned} \tag{2.2}$$

Note that this theorem is **not** limited to irreducible representations.

2.3 Orthogonality relations

2.3.1 Orthogonality of irreducible representations

Theorem: Given two irreducible representations $D^\mu(g)$ and $D^\nu(g)$ of dimensions d_μ and d_ν, the representation matrices fulfil the relation

$$\sum_g D^\dagger_\mu(g)^i{}_j D^\nu(g)^k{}_l = \frac{n_G}{d_\mu}\,\delta^\nu_\mu \delta^i_l \delta^k_j \qquad (D^\dagger_\mu \equiv (D_\mu)^\dagger). \tag{2.3}$$

Proof: For an arbitrary $d_\mu \times d_\nu$ matrix X, define

$$A = \sum_g D^\dagger_\mu(g) X D^\nu(g). \tag{2.4}$$

Then ($\to$ rearrangement lemma),

$$D^\dagger_\mu(g) A D^\nu(g) = D^\dagger_\mu(g)\Big(\sum_{g'} D^\dagger_\mu(g') X D^\nu(g')\Big) D^\nu(g) = \sum_{g'} D^\dagger_\mu(g'g) X D^\nu(g'g) = A. \tag{2.5}$$

Since G is a finite group, the representation matrices can be chosen unitary, $D^\dagger_\mu(g) = (D_\mu)^{-1}(g)$. According to Schur's lemma, we need to distinguish two cases,

- $\mu = \nu$ (i.e. if the representations are equivalent): $A = \lambda \mathbb{1}$, $\lambda \in \mathbb{C}$, or
- $\mu \neq \nu$: $A = 0$.

Choose the matrix X as $(X^k_j)^m{}_n = \delta^m_j \delta^k_n$ for fixed $j = 1, \ldots, d_\mu$ and $k = 1, \ldots, d_\nu$,

$$(A^k_j)^i{}_l = \sum_g D^\dagger_\mu(g)^i{}_m (X^k_j)^m{}_n D^\nu(g)^n{}_l = \sum_g D^\dagger_\mu(g)^i{}_j D^\nu(g)^k{}_l. \tag{2.6}$$

Since $(A^k_j)^i{}_l = 0$ in the case $\mu \neq \nu$, this proves (2.3) for $\mu \neq \nu$. If $\mu = \nu$, taking the trace of

$$(A^k_j)^i{}_l = \lambda^k_j \delta^i_l = \sum_g D^\dagger_\mu(g)^i{}_j D^\mu(g)^k{}_l.$$

gives

$$\lambda^k_j d_\mu = \sum_g (D^\mu(g) D^\dagger_\mu(g))^k{}_j = \sum_g \delta^k_j = n_G \delta^k_j \quad \Rightarrow \quad (A^k_j)^i{}_l = \frac{n_G}{d_\mu}\, \delta^k_j \delta^i_l, \tag{2.7}$$

which proves (2.3) for $\mu = \nu$. #

$\{D^\mu(g_1)^i{}_j, \ldots, D^\mu(g_{n_G})^i{}_j\}$ can be regarded as a vector with n_G components. For each irreducible representation μ there are d_μ^2 such vectors labelled by $i, j = 1, \ldots, d_\mu$. In total, summing over all irreducible representations, there are $\sum_\mu d_\mu^2$ vectors. According to (2.3), these vectors are orthogonal and, hence,

$$\sum_\mu d_\mu^2 \leq n_G, \tag{2.8}$$

because there can be no more than n_G orthogonal vectors with n_G components. In Section 2.3.3 we will show that this is actually an equality.

2.3.2 Orthogonality of characters

Representations are only unique up to similarity transformations ($\widehat{=}$ basis choice).
$\Rightarrow$ Take traces of the representation matrices to obtain relations for characters which are basis independent.

Set $i = j$, $k = l$ in (2.3) and sum over i, k:

$$\begin{aligned} & \sum_g D^\dagger_\mu(g)^i{}_i D^\nu(g)^k{}_k = \frac{n_G}{d_\mu}\, \delta^\nu_\mu \delta^i_k \delta^k_i \\ \sum_{i,k} \quad \Rightarrow \quad & \sum_g \chi^*_\mu(g) \chi^\nu(g) = n_G \delta^\nu_\mu \\ \Leftrightarrow \quad & \sum_{\mathcal{C}} n_{\mathcal{C}} \chi^*_\mu(\mathcal{C}) \chi^\nu(\mathcal{C}) = n_G \delta^\nu_\mu, \end{aligned} \tag{2.9}$$

where $n_{\mathcal{C}}$ is the number of group elements in the class $\mathcal{C}$.

Application: Calculate to which irreducible representations a given (reducible) representation reduces.

The characters $\chi(\mathcal{C})$ of a reducible representaion are given by

$$\chi(\mathcal{C}) = \sum_\mu n_\mu \chi^\mu(\mathcal{C}), \tag{2.10}$$

where n_μ is the number of times the irreducible representation μ appears in the reducible representation.

Calculate n_μ for a given representation:

$$\sum_{\mathcal{C}} n_{\mathcal{C}} \chi^*_\mu(\mathcal{C}) \chi(\mathcal{C}) = \sum_{\mathcal{C}} n_{\mathcal{C}} \sum_\nu n_\nu \chi^*_\mu(\mathcal{C}) \chi^\nu(\mathcal{C}) = \sum_\nu n_\nu n_G \delta^\nu_\mu = n_G n_\mu. \tag{2.11}$$

⇒ Check whether a representation is reducible:

$$\sum_{\mathcal{C}} n_{\mathcal{C}} \chi^*(\mathcal{C})\chi(\mathcal{C}) = \sum_{\mathcal{C}} n_{\mathcal{C}} \sum_{\mu,\nu} n_\mu n_\nu \chi_\mu^*(\mathcal{C})\chi^\nu(\mathcal{C}) = \sum_{\mu,\nu} n_\mu n_\nu n_G \delta_\mu^\nu = n_G \sum_\mu n_\mu^2. \tag{2.12}$$

If this evaluates to n_G, the representation is irreducible, because $\sum_\mu n_\mu^2 = 1$ if all irrededucible representations except one do not appear and one appears once.

2.3.3 Regular representation

The group multiplication can be written as

$$a g_i = g_{a_i} = g_m \delta_{a_i}^m, \qquad a, g_i, g_{a_i} \in G. \tag{2.13}$$

$g_m \delta_{a_i}^m$ is an element of the group ring $\mathbb{C}[G]$.

$\mathbb{C}[G]$ is the set of all complex linear combinations of group elements $\sum_g z_g g$, $z_g \in \mathbb{C}$, $g \in G$. (new structure beyond the group structure!) with product structure derived from the group multiplication (multiplication is distributive wrt. addition).

For $ab = c$, $a, b, c \in G$:

$$abg_i = cg_i \quad \Leftrightarrow \quad g_k \delta_{a_m}^k \delta_{b_i}^m = g_k \delta_{c_i}^k \quad \Rightarrow \quad \delta_{a_m}^k \delta_{b_i}^m = \delta_{c_i}^k, \tag{2.14}$$

which means that the matrices

$$D^{\text{reg}}(g)^i{}_j = \delta_{g_j}^i \tag{2.15}$$

form a representation of G, namely the regular representation.

- For $g \neq e$, $D^{\text{reg}}(g)$ permutes the group elements in a way that leaves no element invariant (rearrangement lemma),
- $D^{\text{reg}}(g)$ is an element of the defining representation of the symmetric group S_{n_G}.
- Characters of the regular representation: $\chi^{\text{reg}}(e) = n_G$, $\chi^{\text{reg}}(g \neq e) = 0$.
- $\sum_\mu n_\mu^2 = n_G$. Proof:

$$\sum_{\mathcal{C}} n_{\mathcal{C}} \chi_{\text{reg}}^*(\mathcal{C})\chi^{\text{reg}}(\mathcal{C}) = (\chi^{\text{reg}}(e))^2 = n_G^2. \tag{2.16}$$

On the other hand, (2.12) gives

$$\sum_{\mathcal{C}} n_{\mathcal{C}} \chi_{\text{reg}}^*(\mathcal{C})\chi^{\text{reg}}(\mathcal{C}) = n_G \sum_\mu n_\mu^2 \quad \Rightarrow \quad \sum_\mu n_\mu^2 = n_G. \qquad \# \tag{2.17}$$

- Each irreducible representation μ appears $n_\mu = d_\mu$ times in the regular representation. Proof:

$$\sum_{\mathcal{C}} n_{\mathcal{C}} \chi^*_\mu(\mathcal{C}) \chi^{\text{reg}}(\mathcal{C}) = \chi^*_\mu(e) \chi^{\text{reg}}(e) = d_\mu n_G. \tag{2.18}$$

On the other hand, (2.11) gives

$$\sum_{\mathcal{C}} n_{\mathcal{C}} \chi^*_\mu(\mathcal{C}) \chi^{\text{reg}}(\mathcal{C}) = n_G n_\mu \quad \Rightarrow \quad n_\mu = d_\mu. \qquad \# \tag{2.19}$$

This also proofs the equality $\sum_\mu d_\mu^2 = n_G$ (cf. Eq. (2.8)), i.e. according to (2.3) there are n_G orthogonal non-vanishing vectors $\{D^\mu(g_1)^i{}_j, \ldots, D^\mu(g_{n_G})^i{}_j\}$ with n_G elements. This is only possible if the set of vectors is complete, hence,

$$\sum_\mu \sum_{i,j=1}^{d_\mu} d_\mu D^\mu(g)^i{}_j D^\dagger_\mu(g')^j{}_i = n_G \delta_{g,g'}. \tag{2.20}$$

The sum of all representation matrices in a class ("class sum") is proportional to $\mathbb{1}$:

$$\mathcal{D}^\mu(\mathcal{C}) = \frac{n_{\mathcal{C}}}{d_\mu} \chi^\mu(\mathcal{C}) \mathbb{1}, \quad \text{where} \quad \mathcal{D}^\mu(\mathcal{C}) = \sum_{h \in \mathcal{C}} D^\mu(h). \tag{2.21}$$

Proof:

$$D^\mu(g) \mathcal{D}^\mu(\mathcal{C}) D^\mu(g)^{-1} = \sum_{h \in \mathcal{C}} D^\mu(\underbrace{ghg^{-1}}_{h' \in \mathcal{C}}) = \sum_{h' \in \mathcal{C}} D^\mu(h') = \mathcal{D}^\mu(\mathcal{C}) \quad \forall g \in G. \tag{2.22}$$

According to Schur's lemma, $\mathcal{D}^\mu(\mathcal{C}) = \lambda^\mu \mathbb{1}$. Take the trace to determine λ^μ:

$$\text{Tr}\{\mathcal{D}^\mu(\mathcal{C})\} = \lambda^\mu \, \text{Tr}\{\mathbb{1}\} \quad \Leftrightarrow \quad n_{\mathcal{C}} \chi^\mu(\mathcal{C}) = \lambda^\mu d_\mu, \tag{2.23}$$

which proofs (2.21). #

Summing (2.20) over group elements $g \in \mathcal{C}$ and $g' \in \mathcal{C}'$ of classes $\mathcal{C}$, $\mathcal{C}'$ proves the completeness of characters:

$$\begin{aligned}
& \sum_{g \in \mathcal{C}} \sum_{g' \in \mathcal{C}'} \sum_\mu \sum_{i,j=1}^{d_\mu} d_\mu D^\mu(g)^i{}_j D^\dagger_\mu(g')^j{}_i = \sum_{g \in \mathcal{C}} \sum_{g' \in \mathcal{C}'} n_G \delta_{g,g'} \\
\Leftrightarrow \quad & \sum_\mu \sum_{i,j=1}^{d_\mu} d_\mu \mathcal{D}^\mu(\mathcal{C}) \mathcal{D}^\dagger_\mu(\mathcal{C}') = n_G n_{\mathcal{C}} \delta_{\mathcal{C},\mathcal{C}'} \\
\Leftrightarrow \quad & \sum_\mu \sum_{i,j=1}^{d_\mu} d_\mu \, \frac{n_{\mathcal{C}}}{d_\mu} \chi^\mu(\mathcal{C}) \delta^i_j \, \frac{n_{\mathcal{C}'}}{d_\mu} \chi^*_\mu(\mathcal{C}') \delta^j_i = n_G n_{\mathcal{C}} \delta_{\mathcal{C},\mathcal{C}'} \\
\Leftrightarrow \quad & n_{\mathcal{C}} \sum_\mu \chi^\mu(\mathcal{C}) \chi^*_\mu(\mathcal{C}') = n_G \delta_{\mathcal{C},\mathcal{C}'}. \qquad \#
\end{aligned} \tag{2.24}$$

2.3.4 Character table

The character table lists the characters of all classes $\mathcal{C}_i$, $i = 1, \ldots, N_c$ (N_c = number of classes) for all irreducible representations μ_r, $r = 1, \ldots, N_R$ (N_R = number of irreducible representations) of a group G.

G	$\mathcal{C}_1 = \{e\}$	$\mathcal{C}_2$	$\ldots$	$\mathcal{C}_{N_c}$
μ_1	$\chi^{\mu_1}(\mathcal{C}_1)$	$\chi^{\mu_1}(\mathcal{C}_2)$	$\ldots$	$\chi^{\mu_1}(\mathcal{C}_{N_c})$
μ_2	$\chi^{\mu_2}(\mathcal{C}_1)$	$\chi^{\mu_2}(\mathcal{C}_2)$	$\ldots$	$\chi^{\mu_2}(\mathcal{C}_{N_c})$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
μ_{N_R}	$\chi^{\mu_{N_R}}(\mathcal{C}_1)$	$\chi^{\mu_{N_R}}(\mathcal{C}_2)$	$\ldots$	$\chi^{\mu_{N_R}}(\mathcal{C}_{N_c})$

Regard all classes as a vector of N_c elements:

The N_R vectors of N_c elements $\{\tilde{\chi}^\mu(\mathcal{C}_1), \ldots, \tilde{\chi}^\mu(\mathcal{C}_{N_c})\}$ of the normalised characters $\tilde{\chi}^\mu(\mathcal{C}) = \sqrt{\frac{n_\mathcal{C}}{n_G}}\chi^\mu(\mathcal{C})$ are orthogonal (2.9) and complete (2.24)

$$\Rightarrow \quad N_R = N_c, \tag{2.25}$$

i.e. the character table is square. In other words, there are always as many inequivalent irreducible representations as there are classes.

Further properties of characters:

- If $\chi^\mu(e) \equiv d_\mu = 1$, then $|\chi^\mu(\mathcal{C})| = 1$ for all classes $\mathcal{C}$.
  Proof: $\chi^\mu(e) = 1$ means that the corresponding representation $D^\mu(g)$ is 1-dimensional $\Rightarrow (D^\mu(g))^* D^\mu(g) = 1 \quad \Rightarrow \quad |\chi^\mu(g)| = |D^\mu(g)| = 1.$ #
- $\chi^\mu(g^{-1}) = (\chi^\mu(g))^*$. In particular, if $g, g^{-1} \in G$, $\chi^\mu(g)$ is real.
  Proof: $D^\mu(g)$ is unitary $\Rightarrow \forall$ eigenvalues λ_k, $k = 1, \ldots, d_\mu$, of $D^\mu(g)$: $|\lambda_k| = 1$.
  $\chi^\mu(g) = \mathrm{Tr}\{D^\mu(g)\} = \sum_k \lambda_k$,
  $\chi^\mu(g^{-1}) = \mathrm{Tr}\{D_\mu^{-1}(g)\} = \sum_k 1/\lambda_k = \sum_k \lambda_k^* = (\chi^\mu(g))^*.$ #

Example: Character table of the quaternionic group Q

The quaternionic group Q is defined by the presentation

$$Q = \langle i, j | i^4 = e, i^2 = j^2, jij^{-1} = i^{-1} \rangle. \tag{2.26}$$

It consists of the 8 elements

$$\{e, \bar{e}, i, \bar{i} \equiv kj, j, \bar{j} \equiv ik, k, \bar{k} \equiv ji\}$$

that satisfy $i^2 = j^2 = k^2 = ijk = \bar{e}$, and $\bar{e}$ commutes with all elements (derive this from the presentation!).

The regular representation decomposes as

$$n_G = 8 = \sum_\mu d_\mu^2 = 1 + 1 + 1 + 1 + 4 \tag{2.27}$$

into four 1-dimensional and one 2-dimensional irreducible representation. The decomposition $8 = 1+1+1+1+1+1+1+1$ is not possible, because Q is not abelian: $ijk = \bar{e} \Rightarrow ij = k \neq \bar{k} = ji$.

$\Rightarrow e$ and $\bar{e}$ are the only elements that commute with all others and $\bar{e}^2 = e$.
$\Rightarrow \mathcal{C}_1 = \{e\}$, $\mathcal{C}_2 = \{\bar{e}\}$, and $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$ must have 2 elements each.
$\bar{k}ik = jiik = \bar{e}i = i^{-1} = \bar{\imath} \Rightarrow \mathcal{C}_3 = \{i, \bar{\imath}\}$, analogously $\mathcal{C}_4 = \{j, \bar{\jmath}\}$, $\mathcal{C}_5 = \{k, \bar{k}\}$.

So far we can tell that the character table has the form

Q	$\mathcal{C}_1 = \{e\}$	$\mathcal{C}_2 = \{\bar{e}\}$	$\mathcal{C}_3 = \{i, \bar{\imath}\}$	$\mathcal{C}_4 = \{j, \bar{\jmath}\}$	$\mathcal{C}_5 = \{k, \bar{k}\}$
$\mu = 1$	1	1	1	1	1
$\mu = 2$	1	$\chi_{2,2}$	$\chi_{2,3}$	$\chi_{2,4}$	$\chi_{2,5}$
$\mu = 3$	1	$\chi_{3,2}$	$\chi_{3,3}$	$\chi_{3,4}$	$\chi_{3,5}$
$\mu = 4$	1	$\chi_{4,2}$	$\chi_{4,3}$	$\chi_{4,4}$	$\chi_{4,5}$
$\mu = 5$	2	$\chi_{5,2}$	$\chi_{5,3}$	$\chi_{5,4}$	$\chi_{5,5}$

Character completeness for $\mathcal{C}_3$:

$$n_{\mathcal{C}_3} \sum_\mu \chi^\mu(\mathcal{C}_3) \chi^\dagger_\mu(\mathcal{C}_3) = 2(1 + |\chi_{2,3}|^2 + |\chi_{3,3}|^2 + |\chi_{4,3}|^2 + |\chi_{5,3}|^2) \stackrel{!}{=} n_G = 8. \tag{2.28}$$

For $\mu = 2, 3, 4$, $|\chi_{\mu,3}| = 1$, because $\chi^\mu(e) = 1 \Rightarrow \chi_{5,3} = 0$.
Analogousy, $\chi_{5,4} = \chi_{5,5} = 0$.

Character orthogonality between $\mu = 1$ and $\mu = 5$:

$$\sum_{\mathcal{C}} n_{\mathcal{C}} \chi^*_1(\mathcal{C}) \chi^5(\mathcal{C}) = 2 + \chi_{5,2} \stackrel{!}{=} 0 \quad \Rightarrow \quad \chi_{5,2} = -2. \tag{2.29}$$

Character orthogonality between $\mu = 2, 3, 4$ and $\mu = 5$ $\Rightarrow$ $\chi_{2,2} = \chi_{3,2} = \chi_{4,2} = 1$.

The remaining characters have $|\chi_{\mu,c}| = 1$, $\mu = 2, 3, 4$, $c = 3, 4, 5$, because $\chi_{\mu,1} = 1$, and must be real, because each class contains the inverses of its elements, hence $\chi_{r,c} = \pm 1$.

Character orthogonality
$\Rightarrow$ for each $\mu = 2, 3, 4$, two of the remaining characters must be -1, one $+1$.

The complete character table is thus

Q	$\{e\}$	$\{\bar{e}\}$	$\{i, \bar{\imath}\}$	$\{j, \bar{\jmath}\}$	$\{k, \bar{k}\}$
$\mu = 1$	1	1	1	1	1
$\mu = 2$	1	1	1	-1	-1
$\mu = 3$	1	1	-1	1	-1
$\mu = 4$	1	1	-1	-1	1
$\mu = 5$	2	-2	0	0	0

Example: Degeneracies in coupled classical harmonic oscillators

System of N point particles of masses m_i, $i = 1, \ldots, N$ at positions $\vec{x}_i$ in d dimensions, coupled by springs of spring constants k_{ij}, $i, j = 1, \ldots, N$, $i > j$.

Lagrangian: $$L = \frac{1}{2}\sum_i m_i \dot{\vec{x}}_i^2 - \frac{1}{2}\sum_{i>j} k_{ij}(\vec{x}_i - \vec{x}_j)^2.$$

Equation of motion can be written as

$$\ddot{x}_a = -K_{ab}x_b, \qquad a = 1, \ldots, Nd \qquad \text{running over all coordinates.} \tag{2.30}$$

Ansatz: $x_a(t) = X_a e^{i\omega t}$.
$\Rightarrow$ Squared eigenfrequencies are given by the eigenvalues of the matrix K.

Symmetry: let the system by invariant under $x \to x' = D(g)x$,
where $D(g)$ is an Nd-dimensional representation of a group G, $g \in G$.
$\Rightarrow$ x' also solves the EOM (2.30) $\Rightarrow$ $D(g)K = KD(g)$.

Use Schur's lemma:

- G has irreducible representations μ of dimension d_μ, $\mu = 1, \ldots$.
- If the (in general reducible) representation $D(g)$ reduces to n_1 times $\mu = 1$, n_2 times $\mu = 2$, ..., then K has the diagonalised form

$$K_{\text{diag}} = \text{diag}\left((\omega_1^{(1)})^2 \mathbb{1}_{d_1}, \ldots, (\omega_1^{(n_1)})^2 \mathbb{1}_{d_1}, (\omega_2^{(1)})^2 \mathbb{1}_{d_2}), \ldots, (\omega_2^{(n_2)})^2 \mathbb{1}_{d_2}, \ldots\right). \tag{2.31}$$

Special case:
$N = 3$ particles of identical mass in $d = 3$ dimensions, coupled by identical springs.
$\Rightarrow$ Symmetry transforms the coordinates under a $Nd = 9$-dimensional representation $D(g)$ of the symmetric group S_3 (rsp. D_3, because $S_3 \simeq D_3$). Need the character table of S_3 (prove this!) and the characters of the representation $D(g)$:

$S_3 \simeq D_3$	$\mathcal{C}_1 = \{e\}$	$\mathcal{C}_2 = \{(123), (132)\}$	$\mathcal{C}_3 = \{(12), (23), (31)\}$
$n_\mathcal{C}$	1	2	3
$\mu = 1$	1	1	1
$\mu = 1'$	1	1	-1
$\mu = 2$	2	-1	0
$D(g)$	9	0	3

The characters of $D(g)$ are easy to find:

- $\chi(e) = \dim(D(g)) = Nd = 9$,
- $\chi(\mathcal{C}_2) = 0$, because the elements of $\mathcal{C}_2$ leave no coordinate invariant,
- $\chi(\mathcal{C}_3) = d = 3$, because the elements of $\mathcal{C}_3$ leave the coordinates of one particle invariant and permutes all others.

Use (2.11) to calculate how often each irreducible representation appears in $D(g)$:

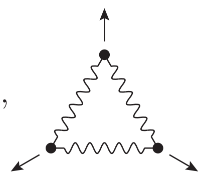
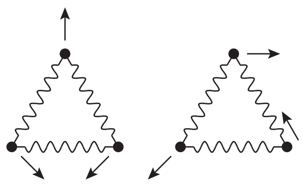
$$n_\mu = \frac{1}{n_G}\sum_{\mathcal{C}} n_{\mathcal{C}}\chi_\mu^*(\mathcal{C})\chi(\mathcal{C}) \quad \Rightarrow \quad \begin{aligned} n_1 &= \frac{1}{6}(1\cdot 1\cdot 9 + 2\cdot 1\cdot 0 + 3\cdot 1\cdot 3) = 3, \\ n_{1'} &= \frac{1}{6}(1\cdot 1\cdot 9 + 2\cdot 1\cdot 0 + 3\cdot(-1)\cdot 3) = 0, \\ n_2 &= \frac{1}{6}(1\cdot 2\cdot 9 + 2\cdot(-1)\cdot 0 + 3\cdot 0\cdot 3) = 3 \end{aligned} \tag{2.32}$$

We expect three 2-fold degeneracies and three non-degenerate modes.
But: This includes the "zero modes", i.e. modes with $\omega = 0$. These are not all symmetry connected by S_3, hence, there are accidental degeneracies (→ space-time symmetries). With some physical intuition, we can identify the modes.

Zero modes ($\omega = 0$):

- 1-dim: translation orthogonal to the plane spanned by the particles,
- 2-dim: translation within the plane,
- 1-dim: rotation around the symmetry axis,
- 2-dim: rotation around the two other axes.

Oscillation modes:

- 1-dim: "breathing mode" 
- 2-dim: two degenerate oscillation modes 

Chapter 3

SO(3) and SU(2)

3.1 The rotation group SO(3)

Definition:

SO(3) $\equiv$ Lie group of all rotations in 3-dim. space.

Defining representation R in 3-dim. vector space $V = \mathbb{R}^3$: $\quad \vec{v} \to \vec{v}\,' = R\vec{v}, \quad \vec{v} \in \mathbb{R}^3$, with the two requirements:

$$\vec{v}^{\,2} \overset{!}{=} \vec{v}\,'^2 = \vec{v}\,'^{\mathrm{T}}\,\vec{v}\,' = \vec{v}^{\,\mathrm{T}} R^{\mathrm{T}} R\,\vec{v}, \qquad R^{\mathrm{T}}R \overset{!}{=} \mathbb{1} \quad (\det R = \pm 1), \tag{3.1}$$

$$\vec{u}\,' \cdot (\vec{v} \times \vec{w}) \overset{!}{=} \vec{u} \cdot (\vec{v}\,' \times \vec{w}\,') = (R\vec{u}) \cdot (R\vec{v} \times R\vec{w}) = \det R \cdot \vec{u} \cdot (\vec{v} \times \vec{w}), \qquad \det R = +1,$$
$$\text{i.e. } R \text{ preserves orientation of 3 vectors.} \tag{3.2}$$

$\Rightarrow$ SO(3) $= \big\{$ 3×3 matrices $R \mid R$ real, $R^{\mathrm{T}}R = \mathbb{1}, \det R = +1\big\}$.

Infinitesimal rotations:

$R = \mathbb{1} + \delta R, \quad \mathbb{1} \overset{!}{=} (\mathbb{1} + \delta R)^{\mathrm{T}}(\mathbb{1} + \delta R) = \mathbb{1} + \delta R + \delta R^{\mathrm{T}} + \cancel{O(\delta R^2)}$,

i.e. $\delta R^{\mathrm{T}} = -\delta R$, antisymmetry.

Note: No restruction on δR from $\det R = 1$, since real orthogonal R with $\det R = -1$ cannot be obtained from $\mathbb{1}$ by continuous deformations.

$$\Rightarrow\; R(\delta\vec{\theta}\,) \equiv \mathbb{1} + \delta R = \begin{pmatrix} 1 & \delta R_{12} & \delta R_{13} \\ & 1 & \delta R_{23} \\ \text{antisym.} & & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & -\delta\theta_3 & \delta\theta_2 \\ & 1 & -\delta\theta_1 \\ \text{antisym.} & & 1 \end{pmatrix} \tag{3.3}$$

$= \mathbb{1} + \delta\vec{\theta}\times, \qquad \delta\theta_a =$ angle for infinitesimal rotation around $\vec{e}_a$ axis

$= \mathbb{1} - \mathrm{i}\delta\vec{\theta} \cdot \vec{J}^{(R)}, \qquad \dim \mathrm{SO}(3) = 3 = \#$ group parameters θ_a.

$\vec{J}^{(R)}$ = generators of SO(3), spanning the Lie algebra so(3) $\equiv$ "angular momentum operator".

↪ 3-dim. "defining representation" R of $\vec{J}$:

$$J_1^{(R)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad J_2^{(R)} = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad J_3^{(R)} = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

Basic commutators of J_a ("Lie algebra") by identifying $J_a \equiv J_a^{(R)}$ in defining repr.:

$$[J_a, J_b] = \mathrm{i} \sum_c \epsilon_{abc} J_c, \qquad \text{verified by explicit calculation, but valid in } all \text{ representations!} \tag{3.5}$$

Specifically, $(J_a^{(R)})_{bc} = -\mathrm{i}\epsilon_{abc}$ is given by the structure constants ϵ_{abc} of so(3) and therefore called "adjoint representation".

Finite rotations:

$$R(\vec{\theta}\,) \;\equiv\; \exp\left\{-\mathrm{i}\vec{\theta}\cdot\vec{J}^{(R)}\right\}, \qquad \vec{\theta} \equiv \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \theta\vec{e} = \text{rotation by angle } \theta \text{ aroung } \vec{e},\ \vec{e}^{\,2} = 1. \tag{3.6}$$

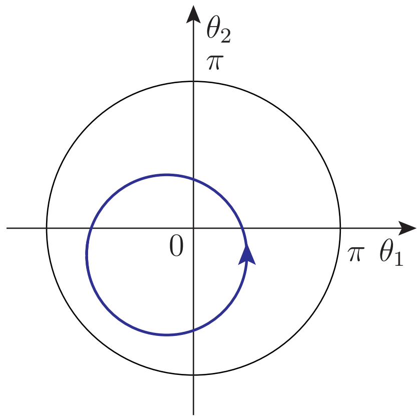
Properties:

- $R(0) = \mathbb{1}$, identity.
- $R(\vec{\theta}\,)$ with $0 < \theta < \pi$ are different for different axes $\vec{e}$, $\vec{e}\,'$.
- $R(\vec{\theta}\,)$ with $\vec{\theta} = \pi\vec{e}$, $\pi\vec{e}\,'$ are different iff $\vec{e}\,' \neq \pm\vec{e}$, i.e. $\pi\vec{e}$ and $-\pi\vec{e}$ are identical.

↪ group parameter space of SO(3)
= sphere of radius π with antipodal points on its surface identified
≡ $\mathbb{R}P^3$ ("real 3-dim. projective space").

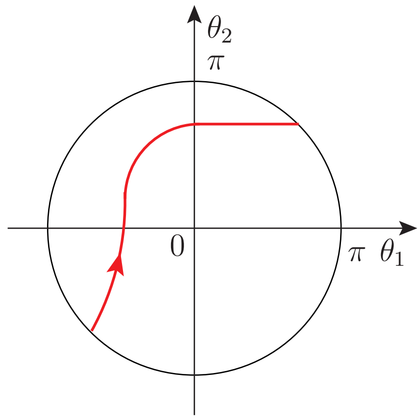
Note: $\mathbb{R}P^3$ is "doubly connected", i.e. ∃ two inequivalent classes of closed curves, where two curves are equivalent ("homotopic") if they can be continuously deformed into each other.

2 examples of inequivalent closed curves $\vec{\theta}(s) \perp \vec{e}_3$ $(0 \leq s \leq 1)$:



$R(\vec{\theta}(s)) \sim R(\vec{0}) = \mathbb{1}$

$\vec{\theta}(s)$ can be deformed into $R(\vec{0}) = \mathbb{1}$.



$R(\vec{\theta}(s)) \nsim R(\vec{0}) = \mathbb{1}$

$\vec{\theta}(s)$ cannot be deformed into $R(\vec{0}) = \mathbb{1}$.

Explicit form of $R(\vec{\theta})$: (straightforward exercise!)

$$R(\vec{\theta}) = \cos\theta \cdot \mathbb{1} + (1-\cos\theta)\, \underbrace{\vec{e}\cdot\vec{e}^{\,\mathrm{T}}}_{=\vec{e}\otimes\vec{e}} + \sin\theta\, \vec{e}\times , \qquad \hookrightarrow \text{ cross product} \tag{3.7}$$

$$R(\vec{\theta})_{ab} = \cos\theta\, \delta_{ab} + (1-\cos\theta)\, e_a e_b - \sin\theta \sum_c \epsilon_{abc} e_c. \tag{3.8}$$

Alternative parametrization via “Euler angles”:

$\hookrightarrow$ Decomposition of rotation around $\vec{\theta}$ into 3 standard rotations:

$$R(\alpha,\beta,\gamma) \equiv \underbrace{R_3(\alpha)\, R_2(\beta)\, R_3(\gamma)}_{R_j(\varphi)\equiv R(\varphi\vec{e}_j) = \text{rotation by angle } \varphi \text{ around } \vec{e}_j} \tag{3.9}$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$0 \le \alpha < 2\pi, \qquad 0 \le \beta < \pi, \qquad 0 \le \gamma < 2\pi.$$

Relation between α, β, γ and $\vec{\theta}$: (straightforward exercise)

$$\cos\theta = \cos\beta\, \cos^2\left(\frac{\alpha+\gamma}{2}\right) - \sin^2\left(\frac{\alpha+\gamma}{2}\right), \tag{3.10}$$

$$e_3 = \frac{\cos^2(\beta/2)\sin(\alpha+\gamma)}{\sin\theta}, \qquad e_1 = \frac{\sin\beta\,(\sin\gamma - \sin\alpha)}{2\sin\theta}, \qquad e_2 = \frac{\sin\beta\,(\cos\alpha + \cos\gamma)}{2\sin\theta}.$$

3.2 The group SU(2)

Definition:
$\mathrm{SU}(2) = \big\{\ 2{\times}2 \text{ matrices } U \mid U \text{ complex},\ U^\dagger U = \mathbb{1},\ \det U = +1\big\}$.

Transformations, generators, Lie algebra:

Parametrization of $U(\vec{\theta}\,)$ by real group parameters $\vec{\theta} = (\theta_1, \ldots, \theta_n)^{\mathrm{T}}$ and generators $\vec{T}$:

$$U(\vec{\theta}\,) = \exp\{-\mathrm{i}\vec{\theta}\cdot\vec{T}\} = \mathbb{1} - \mathrm{i}\vec{\theta}\cdot\vec{T} + \ldots, \tag{3.11}$$

$$U(\vec{\theta}\,)^\dagger = \exp\{\mathrm{i}\vec{\theta}\cdot\vec{T}^\dagger\} = \mathbb{1} + \mathrm{i}\vec{\theta}\cdot\vec{T}^\dagger + \ldots, \tag{3.12}$$

$$U(\vec{\theta}\,)^{-1} = \exp\{\mathrm{i}\vec{\theta}\cdot\vec{T}\}. = \mathbb{1} + \mathrm{i}\vec{\theta}\cdot\vec{T} + \ldots \overset{!}{=} \mathbb{1} + \mathrm{i}\vec{\theta}\cdot\vec{T}^\dagger + \ldots, \tag{3.13}$$

$$\det U(\vec{\theta}\,) = \exp\{-\mathrm{i}\vec{\theta}\cdot\mathrm{Tr}(\vec{T})\} = 1 + \mathrm{i}\vec{\theta}\cdot\mathrm{Tr}(\vec{T}) + \ldots \overset{!}{=} 1. \tag{3.14}$$

$\Rightarrow$ Conditions on 2×2 generators $\vec{T} = (T_1, \ldots, T_n)$:

$$T_a = T_a^\dagger, \qquad \mathrm{Tr}(T_a) = 0. \tag{3.15}$$

$\Rightarrow$ $n = 3$ independent T_a's, usually chosen as $T_a = \frac{1}{2}\sigma_a$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{"Pauli matrices".} \tag{3.16}$$

Lie algebra su(2) = so(3) (by explicit calculation):

$$[T_a, T_b] = \mathrm{i}\sum_c \epsilon_{abc} T_c, \tag{3.17}$$

Note: $\mathrm{su}(2) = \mathrm{so}(3) \equiv \{\sum_a c_a T_a \mid c_a \in \mathbb{R}\}$ = 3-dim. Lie algebra over $\mathbb{R}$,
$\mathrm{sl}(2) \equiv \{\sum_a c_a T_a \mid c_a \in \mathbb{C}\}$, = 3-dim. Lie algebra over $\mathbb{C}$.

Finite group transformations:

$$U(\vec{\theta}\,) = \cos\tfrac{\theta}{2}\cdot\mathbb{1} - \mathrm{i}\sin\tfrac{\theta}{2}\,(\vec{e}\cdot\vec{\sigma}), \qquad \vec{\theta} = \theta\,\vec{e}, \tag{3.18}$$

$$SU(2) = \big\{U(\vec{\theta}\,)\,|\,0 \le \theta \le 2\pi,\ \vec{e} \in S^2 = \text{unit sphere in } \mathbb{R}^3\big\}. \tag{3.19}$$

$\hookrightarrow$ Group parameter space = compact ball $B_{2\pi}$ of radius 2π in $\mathbb{R}^3$ (singly connected).

Relation between SU(2) and SO(3):

- su(2) = so(3) $\Rightarrow$ SU(2) and SO(3) are locally isomorphic.
- But: SU(2) and SO(3) are *not* fully isomorphic, since group parameter spaces are not isomorphic (connectedness!).
- Precise relation obtained by inspecting the group homomorphism

$$f : \mathrm{SU}(2) \to \mathrm{SO}(3), \qquad f\left(U(\vec{\theta}\,)\right) = R(\vec{\theta}\,), \qquad \vec{\theta} \in B_{2\pi}. \tag{3.20}$$

  Determine kernel of f: $R(\vec{\theta}\,) = \mathbb{1}_3 \Leftrightarrow \theta = 0 \vee 2\pi \Leftrightarrow U = \pm\mathbb{1}$.
  $\hookrightarrow$ $\ker(f) = \{\pm\mathbb{1}\} \simeq \mathbb{Z}_2$.
  $\Rightarrow$ $\mathrm{SO}(3) \simeq \mathrm{SU}(2)/\mathbb{Z}_2$ according to first isomorphism theorem (Section 1.3.3).
  Correspondence: $R \leftrightarrow \{U, -U\}$,
  i.e. SO(3) is multivalued on $B_{2\pi}$ and SU(2) doubly covers SO(3).
  SU(2) = "universal covering group" (simply connected) of SO(3).

- Implication on representations:
  - Each representation of SO(3) defines a repr. of SU(2), where $D(2\pi\vec{e}) = \mathbb{1}$.
  - Only representations of SU(2) with $D(2\pi\vec{e}) = \mathbb{1}$ define reprs. of SO(3).
  - Representations of SU(2) with $D(2\pi\vec{e}) = -\mathbb{1}$ define "ray (or projective) representations" of SO(3), which define $D(g)$ for $g \in G$ only up to some constant:

$$D(g)\, D(g') \;\propto\; D(gg').$$

Comment: Ray representations are "good enough" to describe symmetries in QM, because qm. states are "rays" (=states with arbitrary normalization and phases) in some Hilbert space.
SO(3): group of rotations in classical physics,
SU(2): group describing rotations in QM.

3.3 Irreducible representations of SU(2) and SO(3)

Irred. representations of su(2) and so(3):

$\hookrightarrow$ known from eigenvalue problem of angular momentum in QM:

For each $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ $\exists$ $(2j+1)$ simultaneous eigenstates $\{|j,m\rangle \,|\, m = -j, \ldots, j\}$ of J_3 and $\vec{J}^2$, which span some $(2j+1)$-dim. vector space $V^{(j)}$:

$$\begin{aligned} J_3\,|j,m\rangle &= m\,|j,m\rangle, \\ \vec{J}^2\,|j,m\rangle &= j(j+1)\,|j,m\rangle, \\ J_+\,|j,m\rangle &= \sqrt{j(j+1)-m(m+1)}\,|j,m+1\rangle, \\ J_-\,|j,m\rangle &= \sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle, \end{aligned} \tag{3.21}$$

with the "shift operators" $J_\pm = J_1 \pm \mathrm{i} J_2$ obeying

$$[J_3, J_\pm] \;=\; \pm J_\pm, \qquad [J_+, J_-] \;=\; 2J_3. \tag{3.22}$$

Note: $\vec{J}^2$ = "Casimir operator", i.e. $[\vec{J}^2, J_a] = 0$, but $\vec{J}^2 \notin \mathrm{su}(2)$.

$\Rightarrow$ Each j defines a $(2j+1)$-dim. representation $D^{(j)}$:

$$|j,j\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \qquad |j,j-1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \qquad \ldots \qquad |j,-j\rangle = \begin{pmatrix} \vdots \\ 0 \\ 1 \end{pmatrix},$$

$$J_3^{(j)} = \mathrm{diag}(j, j-1, \ldots, -j), \qquad \left(\vec{J}^{(j)}\right)^2 = j(j+1)\,\mathbb{1},$$

$$J_+^{(j)} = \begin{pmatrix} 0 & * & 0 & \ldots & 0 \\ & 0 & * & & \\ \vdots & & 0 & \ddots & \vdots \\ & & & \ddots & * \\ 0 & & \ldots & & 0 \end{pmatrix}, \qquad J_-^{(j)} = \left(J_+^{(j)}\right)^\dagger = \begin{pmatrix} 0 & & \ldots & & 0 \\ * & 0 & & & \\ 0 & * & 0 & \ddots & \vdots \\ \vdots & & & \ddots & \\ 0 & & \ldots & * & 0 \end{pmatrix}. \tag{3.23}$$

Features of $D^{(j)}$:

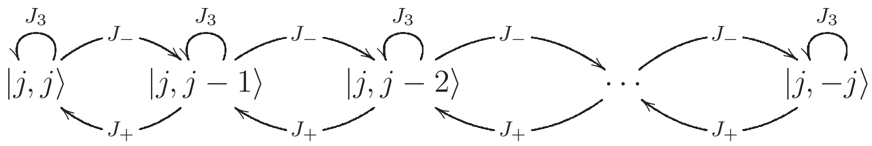
- Consider su(2) as vector space spanned by basis $\{J_3, J_+, J_-\}$.
  - $\hookrightarrow$ Brackets $[J_a, X] \in \mathrm{su}(2)$ act as linear operator (matrices!) on $X \in \mathrm{su}(2)$.
  - $\hookrightarrow$ The matrices $\mathrm{ad}_{J_a} \equiv [J_a, \,.\,]$ define a 3-dim. repr. of su(2) on the vector space su(2), which is identical with the adjoint prepresentation:

$$[\mathrm{ad}_{J_a}, \mathrm{ad}_{J_b}] \;=\; \sum_c \mathrm{i}\epsilon_{abc}\,\mathrm{ad}_{J_c} \tag{3.24}$$

  Note: The basis $\{J_3, J_+, J_-\}$ is very special:

  - J_3 is diagonal: $\mathrm{ad}_{J_3}(X) = [J_3, X] = f(X)\,X$.
  - $J_\pm$ are nilpotent: $\mathrm{ad}^3_{J_\pm}(X) = [J_\pm, [J_\pm, [J_\pm, X]]] = 0$.

- Irreducibility:



All basis states $|j,m\rangle$ can be obtained from a single state upon applying $(J_\pm)^n$, e.g.

$$\underbrace{|j,m\rangle}_{\text{state of “weight” } m} \;\propto\; \left(J_-^{(j)}\right)^{m-j} \underbrace{|j,j\rangle}_{\text{state of “maximal weight”}}, \qquad \left(J_+^{(j)}\right)|j,j\rangle \;=\; 0. \tag{3.25}$$

Example: $j = 1$.

- Generators:

$$J_3^{(1)} = \mathrm{diag}(1,0,-1), \qquad \left(\vec{J}^{(1)}\right)^2 = 2\cdot \mathbb{1},$$

$$J_+^{(1)} = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad J_-^{(1)} = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$J_1^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}. \tag{3.26}$$

- Relation to 3-dim. defining representation R of so(3):

$$J_1^{(R)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad J_2^{(R)} = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad J_3^{(R)} = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.27}$$

Check whether $D^{(1)}$ and R are equivalent:

$$J_a^{(R)} \overset{?}{=} S\, J_a^{(1)}\, S^{-1}. \tag{3.28}$$

1. Diagonalize $J_3^{(R)}$.
   $\hookrightarrow$ $S = (\vec{n}_1, \vec{n}_2, \vec{n}_3)$, $\quad \vec{n}_a$ = eigenvectors of $J_3^{(R)}$,

$$\vec{n}_1 = \frac{\mathrm{e}^{\mathrm{i}\delta_1}}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i} \\ 0 \end{pmatrix}, \qquad \vec{n}_2 = \mathrm{e}^{\mathrm{i}\delta_2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{n}_3 = \frac{\mathrm{e}^{\mathrm{i}\delta_3}}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \\ 0 \end{pmatrix}. \tag{3.29}$$

2. Check whether phases δ_a can be chosen so that (3.28) is valid for $a = 1, 2$.
   $\hookrightarrow$ Answer: yes! $\quad 1 = -\mathrm{e}^{\mathrm{i}\delta_1} = \mathrm{e}^{\mathrm{i}\delta_2} = \mathrm{e}^{\mathrm{i}\delta_3}$.

$$\Rightarrow\; S \;=\; \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -\mathrm{i}/\sqrt{2} & 0 & -\mathrm{i}/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.30}$$

$\Rightarrow$ (3.28) holds, i.e. $R \simeq D^{(1)}$.

Irred. representations of SU(2) and SO(3):

$\hookrightarrow$ obtained from $D^{(j)}$ representation of the generators J_a:

$$D^{(j)}(\vec{\theta}) \equiv \exp\{-\mathrm{i}\vec{\theta}\,\vec{J}^{(j)}\} = (2j+1)\times(2j+1) \text{ matrix} \tag{3.31}$$

$$D^{(j)}(\vec{\theta})_{m'm} = \langle j,m'|\,\exp\{-\mathrm{i}\vec{\theta}\,\vec{J}\}\,|j,m\rangle. \tag{3.32}$$

Here Euler angles are particularly convenient:

$$\begin{aligned} D^{(j)}(\alpha,\beta,\gamma)_{m'm} &= \langle j,m'|\,\exp\{-\mathrm{i}\alpha\,J_3^{(j)}\}\,\exp\{-\mathrm{i}\beta\,J_2^{(j)}\}\,\exp\{-\mathrm{i}\gamma\,J_3^{(j)}\}\,|j,m\rangle \\ &= \mathrm{e}^{-\mathrm{i}m'\alpha-\mathrm{i}m\gamma}\,\underbrace{\langle j,m'|\,\exp\{-\mathrm{i}\beta\,J_2\}\,|j,m\rangle}_{\equiv d^{(j)}_{m'm}(\beta),\ \text{“Wigner's } d\text{-functions}}. \end{aligned} \tag{3.33}$$

Properties:

- Irreducibility of $D^{(j)}$ follows from irreducibility of $J_a^{(j)}$.
- Explicit closed form:

$$\begin{aligned} d^{(j)}_{m'm}(\beta) = \sum_k (-1)^{k-m+m'} & \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!k!(j-k-m')!(k-m+m')!} \\ \uparrow \qquad & \times \left(\cos\tfrac{\beta}{2}\right)^{2j-2k+m-m'}\left(\sin\tfrac{\beta}{2}\right)^{2k-m+m'}, \end{aligned} \tag{3.34}$$

all $k\in\mathbb{N}_0$ with $k\le j+m$, $k\le j-m'$, $k\ge m-m'$.

  Possible proofs are based on:

  - $d(\beta)$ as normalizable solutions of the differential eq.

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\beta^2}+\cot\beta\,\frac{\mathrm{d}}{\mathrm{d}\beta}-\frac{m^2+m'^2-2mm'\cos\beta}{\sin^2\beta}+j(j+1)\right]d(\beta)=0, \tag{3.35}$$

    which is related to the Jacobi differential eq.
  - Analysis of “Schwinger's oscillator model” of angular momentum.

- $D^{(j)}(\alpha,\beta,\gamma)$ = unitary matrix,
  $d^{(j)}_{m'm}(\beta)$ = real orthogonal matrix (clever choice of Euler rotations!).
- Symmetries: $d^{(j)}_{m'm}(\beta) = (-1)^{m-m'}\,d^{(j)}_{mm'}(\beta) = (-1)^{m-m'}\,d^{(j)}_{-m',-m}(\beta)$.
- Orthogonality:

$$\begin{aligned} &\underbrace{\int_0^{2\pi}\mathrm{d}\alpha\int_0^{\pi}\mathrm{d}\beta\,\sin\beta\int_0^{2\pi}\mathrm{d}\gamma}_{\text{Haar measure of SU(2)}}\,D^{(j_1)}_{m_1'm_1}(\alpha,\beta,\gamma)^*\,D^{(j_2)}_{m_2'm_2}(\alpha,\beta,\gamma) \\ &\qquad = \frac{8\pi^2}{2j_1+1}\,\delta_{j_1j_2}\,\delta_{m_1m_2}\,\delta_{m_1'm_2'}. \end{aligned} \tag{3.36}$$

- Global properties and action on states $|\psi\rangle \in V^{(j)}$:

representation for	$j = 0, 1, 2, \ldots$	$j = \frac{1}{2}, \frac{3}{2}, \ldots$
$D^{(j)}(\vec{\theta})$ in SO(3)	single valued	double valued
$D^{(j)}(\vec{\theta})$ in SU(2)	single valued	single valued
$D^{(j)}(2\pi\vec{e})\,|\psi\rangle =$	$+|\psi\rangle$	$-|\psi\rangle$
$D^{(j)}(4\pi\vec{e})\,|\psi\rangle =$	$+|\psi\rangle$	$+|\psi\rangle$
state =	bosonic	fermionic

3.4 Product representations and Clebsch–Gordan decomposition

Qm. problem of addition of angular momenta:

Consider a qm. system of 2 independent components (e.g. 2 particles) with angular momenta $\vec{J}_k$ $(k = 1, 2)$ each, i.e.

$$\begin{aligned} \vec{J}_k^{\,2}\,|j_k, m_k\rangle &= j_k(j_k+1)\,|j_k, m_k\rangle, && j_k = 0, \tfrac{1}{2}, 1, \cdots = \text{fixed!} \\ J_{k,3}\,|j_k, m_k\rangle &= m_k\,|j_k, m_k\rangle, && m_k = -j_k, -j_k+1, \ldots, j_k, \\ [J_{1,a}, J_{2,b}] &= 0, \text{ independence of 2 components!} \end{aligned} \tag{3.37}$$

$\Rightarrow$ Product basis of Hilbert space $\mathcal{H}$: $|j_1, j_2; m_1, m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle$.
$\hookrightarrow$ $(2j_1+1)(2j_2+1)$ states

Problem:
Express eigenstates $|j, m\rangle$ of total angular momentum $\vec{J} = \vec{J}_1 + \vec{J}_2 (\equiv \vec{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{J}_2)$

$$\begin{aligned} \vec{J}^{\,2}\,|j, m\rangle &= j(j+1)\,|j, m\rangle, && j = ? \\ J_3\,|j, m\rangle &= m\,|j, m\rangle, && m = -j, -j+1, \ldots, j \end{aligned} \tag{3.38}$$

in terms of $|j_1, j_2; m_1, m_2\rangle$!

Commutators:

$$[J_a, J_b] = \mathrm{i} \sum_c \epsilon_{abc}\, J_c, \quad \text{since } \vec{J} = \vec{J}_1 + \vec{J}_2, \ [J_{1,a}, J_{2,b}] = 0. \tag{3.39}$$

$\hookrightarrow$ $\vec{J} =$ indeed angular momentum operator.

$$\left.\begin{aligned} [J_3, J_{k,3}] = 0, \quad [J_3, \vec{J}_k^{\,2}] = 0, \\ [\vec{J}^{\,2}, J_{k,3}] \neq 0, \quad [\vec{J}^{\,2}, \vec{J}_k^{\,2}] = 0, \end{aligned}\right\} \begin{aligned} &\text{Simultaneously diagonalizable:} \quad \vec{J}_1^{\,2}, \vec{J}_2^{\,2}, \vec{J}^{\,2}, J_3. \\ &\hookrightarrow \text{ Eigenstates: } |j, m\rangle \equiv |j_1, j_2, j, m\rangle. \end{aligned} \tag{3.40}$$

Basis change:

$$|j, m\rangle = \sum_{\substack{j_1', j_2', \\ m_{1,2}}} |j_1', j_2'; m_1, m_2\rangle \underbrace{\langle j_1', j_2'; m_1, m_2 | j, m\rangle}_{\text{“Clebsch–Gordan coefficients”}} \tag{3.41}$$

$\neq 0$ only if $j_1' = j_1$, $j_2' = j_2$,

$$\begin{aligned} \text{because } 0 &= \langle j_1', j_2'; m_1, m_2|\, \vec{J}_k^{\,2} - \vec{J}_k^{\,2}\, |j_1, j_2, j, m\rangle \\ &= \underbrace{[j_k'(j_k'+1) - j_k(j_k+1)]}_{\neq 0 \text{ for } j_k' \neq j_k} \langle j_1', j_2'; m_1, m_2 | j_1, j_2, j, m\rangle. \end{aligned}$$

$$\Rightarrow \ |j, m\rangle = \sum_{m_{1,2}} |j_1, j_2; m_1, m_2\rangle \underbrace{\langle j_1, j_2; m_1, m_2 | j, m\rangle}_{\neq 0 \text{ only if } m = m_1 + m_2,} \tag{3.42}$$

$$\begin{aligned} \text{because } 0 &= \langle j_1, j_2; m_1, m_2|\, J_{1,3} + J_{2,3} - J_3\, |j_1, j_2, j, m\rangle \\ &= (m_1 + m_2 - m)\, \langle j_1, j_2; m_1, m_2 | j_1, j_2, j, m\rangle. \end{aligned}$$

Note: Both $\{|j_1, j_2; m_1, m_2\rangle\}$ and $\{|j, m\rangle\}$ are orthonormal bases!
$\Rightarrow$ Orthogonality relations:

$$\sum_{j,m} \langle j_1, j_2; m_1, m_2|j, m\rangle \, \langle j, m|j_1, j_2; m_1', m_2'\rangle = \delta_{m_1 m_1'} \, \delta_{m_2 m_2'}, \tag{3.43}$$

$$\sum_{m_1,m_2} \langle j, m|j_1, j_2; m_1, m_2\rangle \, \langle j_1, j_2; m_1, m_2|j', m'\rangle = \delta_{jj'} \, \delta_{mm'}. \tag{3.44}$$

Calculation of CG coefficients:

- Step 0: $m = m_{\max}$.

$$m_{\max} = \max(m_1 + m_2) = j_1 + j_2. \quad \Rightarrow \; j_{\max} = j_1 + j_2. \tag{3.45}$$

$$|j = j_1 + j_2, m = j_1 + j_2\rangle \equiv |j_1, j_2; j_1, j_2\rangle, \quad \text{unique up to phase choice!} \tag{3.46}$$

$$\Rightarrow \langle j_1, j_2; j_1, j_2|j_1 + j_2, j_1 + j_2\rangle = 1. \tag{3.47}$$

- Step 1: $m = m_{\max} - 1$.
Application of $J_- \, |j, m\rangle = \sqrt{j(j+1) - m(m-1)} \, |j, m-1\rangle$:

$$\begin{aligned} J_- \, |j_1 + j_2, j_1 + j_2\rangle &= \sqrt{2(j_1 + j_2)} \, |j_1 + j_2, j_1 + j_2 - 1\rangle \\ &= (J_{1-} + J_{2-}) \, |j_1, j_2; j_1, j_2\rangle \\ &= \sqrt{2j_1} \, |j_1, j_2; j_1 - 1, j_2\rangle + \sqrt{2j_2} \, |j_1, j_2; j_1, j_2 - 1\rangle, \end{aligned} \tag{3.48}$$

$$|j_1 + j_2, j_1 + j_2 - 1\rangle = \sqrt{\frac{j_1}{j_1 + j_2}} \, |j_1, j_2; j_1 - 1, j_2\rangle + \sqrt{\frac{j_2}{j_1 + j_2}} \, |j_1, j_2; j_1, j_2 - 1\rangle. \tag{3.49}$$

$$\begin{aligned} \Rightarrow \; \langle j_1, j_2; j_1 - 1, j_2|j_1 + j_2, j_1 + j_2 - 1\rangle &= \sqrt{\frac{j_1}{j_1 + j_2}}, \\ \langle j_1, j_2; j_1, j_2 - 1|j_1 + j_2, j_1 + j_2 - 1\rangle &= \sqrt{\frac{j_2}{j_1 + j_2}}. \end{aligned} \tag{3.50}$$

$\exists$ (2nd state with $m = j_1 + j_2 - 1$) $\perp$ $|j_1 + j_2, j_1 + j_2 - 1\rangle$:

$$\underbrace{|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle}_{\text{Check eigenvalue of } \vec{J}^2 \text{ explicitly!}} = \sqrt{\frac{j_2}{j_1 + j_2}} \, |j_1, j_2; j_1 - 1, j_2\rangle - \sqrt{\frac{j_1}{j_1 + j_2}} \, |j_1, j_2; j_1, j_2 - 1\rangle. \tag{3.51}$$

$\nwarrow$ phase choice!

$$\begin{aligned} \Rightarrow \; \langle j_1, j_2; j_1 - 1, j_2|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle &= \sqrt{\frac{j_2}{j_1 + j_2}}, \\ \langle j_1, j_2; j_1, j_2 - 1|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle &= -\sqrt{\frac{j_1}{j_1 + j_2}}. \end{aligned} \tag{3.52}$$

- Step 2: $m = m_{\max} - 2$.

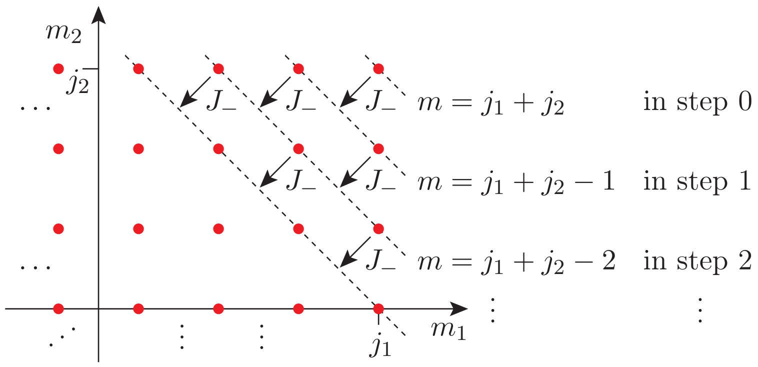
  Construct 3 states:

$$J_- |j_1 + j_2, j_1 + j_2 - 1\rangle \propto |j_1 + j_2, j_1 + j_2 - 2\rangle = \dots \quad (3.53)$$
$$J_- |j_1 + j_2 - 1, j_1 + j_2 - 1\rangle \propto |j_1 + j_2 - 1, j_1 + j_2 - 2\rangle = \dots \quad (3.54)$$
$$\text{via orthogonalization:} \quad |j_1 + j_2 - 1, j_1 + j_2 - 2\rangle. \quad (3.55)$$

  $\hookrightarrow$ Express them in terms of $|j_1, j_2; j_1 - 2, j_2\rangle$,
  $|j_1, j_2; j_1 - 1, j_2 - 1\rangle$,
  $|j_1, j_2; j_1, j_2 - 2\rangle$.

  $\Rightarrow$ 9 CG coefficients with $m = j_1 + j_2 - 2$.

  Graphical illustration:



- Step k: $m = m_{\max} - k$.

  Construct $k + 1$ states:

$$J_- |j_1 + j_2, j_1 + j_2 - k + 1\rangle \propto |j_1 + j_2, j_1 + j_2 - k\rangle = \dots \quad (3.56)$$
$$\vdots \qquad \vdots$$
$$J_- |j_1 + j_2 - k + 1, j_1 + j_2 - k + 1\rangle \propto |j_1 + j_2 - k + 1, j_1 + j_2 - k\rangle = \dots \quad (3.57)$$
$$\text{via orthogonalization:} \quad |j_1 + j_2 - k, j_1 + j_2 - k\rangle. \quad (3.58)$$

  But: $m_{1,2}$ values:

$$m_1 = j_1 - k \overset{!}{\geq} -j_1, \qquad m_2 = j_2$$
$$\vdots \qquad \vdots$$
$$m_1 = j_1, \qquad m_2 = j_2 - k \overset{!}{\geq} -j_2.$$
$$\Rightarrow \quad k \leq \min(2j_1, 2j_2).$$

  Otherwise there cannot be a new state with $j = j_1 + j_2 - k$!

$$\Rightarrow \quad j_{\min} = j_1 + j_2 - \min(2j_1, 2j_2) = |j_1 - j_2|. \quad (3.59)$$

- Further steps analogously until $m = -m_{\max} = m_{\min}$, but no new states via orthogonalization for $m < |j_1 - j_2|$.

$$\#\ \text{states} = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = \sum_{j=0}^{j_1+j_2} 2j - \sum_{j=0}^{|j_1-j_2|-1} 2j + j_1 + j_2 - \big(|j_1 - j_2| - 1\big)$$
$$= (j_1 + j_2)(j_1 + j_2 - 1) - \big(|j_1 - j_2| - 1\big)|j_1 - j_2| + j_1 + j_2 - \big(|j_1 - j_2| - 1\big)$$
$$= (2j_1 + 1)(2j_2 + 1). \qquad \#$$

Example: $j_1 = \frac{1}{2},\ j_2 = 1. \quad \Rightarrow\ j = \frac{3}{2}, \frac{1}{2}.$

Bases:

$$||m_1, m_2\rangle\rangle \equiv |\tfrac{1}{2}, 1; m_1, m_2\rangle : \qquad m_1 = \pm\tfrac{1}{2},\ m_2 = 0, \pm 1,$$

$$|j, m\rangle : \qquad j = \tfrac{3}{2},\ m = \pm\tfrac{3}{2}, \pm\tfrac{1}{2};$$
$$j = \tfrac{1}{2},\ m = \pm\tfrac{1}{2}.$$

Construction of states:

$$m = \tfrac{3}{2}: \qquad |\tfrac{3}{2}, \tfrac{3}{2}\rangle = ||\tfrac{1}{2}, 1\rangle\rangle, \qquad \text{highest-weight state.} \tag{3.60}$$

$$\begin{aligned} m = \tfrac{1}{2}: \qquad J_- |\tfrac{3}{2}, \tfrac{3}{2}\rangle &= \sqrt{3}\, |\tfrac{3}{2}, \tfrac{1}{2}\rangle \\ &= J_{1-}\, ||\tfrac{1}{2}, 1\rangle\rangle + J_{2-}\, ||\tfrac{1}{2}, 1\rangle\rangle = ||-\tfrac{1}{2}, 1\rangle\rangle + \sqrt{2}\, ||\tfrac{1}{2}, 0\rangle\rangle, \end{aligned}$$

$$\Rightarrow\ |\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, ||-\tfrac{1}{2}, 1\rangle\rangle + \sqrt{\tfrac{2}{3}}\, ||\tfrac{1}{2}, 0\rangle\rangle, \tag{3.61}$$

$$\Rightarrow\ |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, ||-\tfrac{1}{2}, 1\rangle\rangle - \sqrt{\tfrac{1}{3}}\, ||\tfrac{1}{2}, 0\rangle\rangle. \tag{3.62}$$

$$\begin{aligned} m = -\tfrac{1}{2}: \qquad J_- |\tfrac{3}{2}, \tfrac{1}{2}\rangle &= 2\, |\tfrac{3}{2}, -\tfrac{1}{2}\rangle \\ &= \sqrt{\tfrac{1}{3}}\, (J_{1-} + J_{2-})\, ||-\tfrac{1}{2}, 1\rangle\rangle + \sqrt{\tfrac{2}{3}}\, (J_{1-} + J_{2-})\, ||\tfrac{1}{2}, 0\rangle\rangle \\ &= \sqrt{\tfrac{2}{3}}\, ||-\tfrac{1}{2}, 0\rangle\rangle + \sqrt{\tfrac{2}{3}}\, ||-\tfrac{1}{2}, 0\rangle\rangle + \sqrt{\tfrac{4}{3}}\, ||\tfrac{1}{2}, -1\rangle\rangle, \end{aligned}$$

$$\Rightarrow\ |\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, ||-\tfrac{1}{2}, 0\rangle\rangle + \sqrt{\tfrac{1}{3}}\, ||\tfrac{1}{2}, -1\rangle\rangle, \tag{3.63}$$

$$\begin{aligned} J_- |\tfrac{1}{2}, \tfrac{1}{2}\rangle &= |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \\ &= \sqrt{\tfrac{2}{3}}\, (J_{1-} + J_{2-})\, ||-\tfrac{1}{2}, 1\rangle\rangle - \sqrt{\tfrac{1}{3}}\, (J_{1-} + J_{2-})\, ||\tfrac{1}{2}, 0\rangle\rangle \\ &= \sqrt{\tfrac{4}{3}}\, ||-\tfrac{1}{2}, 0\rangle\rangle - \sqrt{\tfrac{1}{3}}\, ||-\tfrac{1}{2}, 0\rangle\rangle - \sqrt{\tfrac{2}{3}}\, ||\tfrac{1}{2}, -1\rangle\rangle. \end{aligned}$$

$$\Rightarrow\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, ||-\tfrac{1}{2}, 0\rangle\rangle - \sqrt{\tfrac{2}{3}}\, ||\tfrac{1}{2}, -1\rangle\rangle. \tag{3.64}$$

$$\begin{aligned} m = -\tfrac{3}{2}: \qquad J_- |\tfrac{3}{2}, -\tfrac{1}{2}\rangle &= \sqrt{3}\, |\tfrac{3}{2}, -\tfrac{3}{2}\rangle \\ &= \sqrt{\tfrac{2}{3}}\, (J_{1-} + J_{2-})\, ||-\tfrac{1}{2}, 0\rangle\rangle + \sqrt{\tfrac{1}{3}}\, (J_{1-} + J_{2-})\, ||\tfrac{1}{2}, -1\rangle\rangle \\ &= \sqrt{\tfrac{4}{3}}\, ||-\tfrac{1}{2}, -1\rangle\rangle + \sqrt{\tfrac{1}{3}}\, ||-\tfrac{1}{2}, -1\rangle\rangle \end{aligned}$$

$$\Rightarrow\ |\tfrac{3}{2}, -\tfrac{3}{2}\rangle = ||-\tfrac{1}{2}, -1\rangle\rangle. \tag{3.65}$$

Clebsch–Gordan series:

$$|j,m\rangle = \sum_{\substack{m_1 \\ (m_2=m-m_1)}} |j_1,j_2;m_1,m_2\rangle \underbrace{\langle j_1,j_2;m_1,m_2|j,m\rangle}_{\equiv C^{(j)}_{m_1 m},} . \tag{3.66}$$

$$C = C^{(j_1+j_2)} \oplus \cdots \oplus C^{(|j_1-j_2|)} = \text{unitary}$$

$$\Rightarrow \langle j,m'|\, A\, |j,m\rangle = \sum_{\substack{m_1 \\ (m_2=m-m_1)}} \langle j,m'|\, A\, |j_1,j_2;m_1,m_2\rangle\, C^{(j)}_{m_1 m}$$

$$= \sum_{\substack{m_1,m_1' \\ (m_2=m-m_1 \\ m_2'=m'-m_1')}} C^{(j)\,*}_{m_1' m'}\, \langle j_1,j_2;m_1',m_2'|\, A\, |j_1,j_2;m_1,m_2\rangle\, C^{(j)}_{m_1 m} \tag{3.67}$$

Matrix notation:

$$A^{(j)} = C^{(j)\,\dagger}\, A^{(j_1\otimes j_2)}\, C^{(j)}, \qquad |j,j\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \quad |j,j-1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \quad \text{etc.} \tag{3.68}$$

Block structure of $\oplus_j A^{(j)} = \oplus_j \big(\vec{J}^{(j)}\big)^2,\ \oplus_j J_3^{(j)}, \oplus_j J_\pm^{(j)}$: $\qquad (j_{\max} = j_1{+}j_2,\ j_{\min} = |j_1{-}j_2|)$

$$\oplus_{j=j_{\min}}^{j_{\max}} \big(\vec{J}^{(j)}\big)^2 = \begin{pmatrix} \big(\vec{J}^{(j_{\max})}\big)^2 & & & \\ & \big(\vec{J}^{(j_{\max}-1)}\big)^2 & & \\ & & \ddots & \\ & & & \big(\vec{J}^{(j_{\min})}\big)^2 \end{pmatrix}, \quad \big(\vec{J}^{(j)}\big)^2 = j(j+1)\cdot \mathbb{1}_{2j+1},$$

$$= \text{diagonal},$$

$$\oplus_{j=j_{\min}}^{j_{\max}} J_3^{(j)} = \begin{pmatrix} J_3^{(j_{\max})} & & & \\ & J_3^{(j_{\max}-1)} & & \\ & & \ddots & \\ & & & J_3^{(j_{\min})} \end{pmatrix}, \quad J_3^{(j)} = \mathrm{diag}(j, j-1, \ldots, -j),$$

$$= \text{diagonal},$$

$$\oplus_{j=j_{\min}}^{j_{\max}} J_\pm^{(j)} = \begin{pmatrix} J_\pm^{(j_{\max})} & & & \\ & J_\pm^{(j_{\max}-1)} & & \\ & & \ddots & \\ & & & J_\pm^{(j_{\min})} \end{pmatrix}, \quad J_\pm^{(j)} = (2j+1)\times(2j+1)\ \text{matrix},$$

$$= \text{block-diagonal}. \tag{3.69}$$

$\Rightarrow$ CG decomposition of $D^{(j_1)} \otimes D^{(j_2)}$:

$$C^\dagger \left[D^{(j_1)} \otimes D^{(j_2)}\right] C = \oplus_{j=j_{\min}}^{j_{\max}} D^{(j)}, \quad D^{(j)} = \text{irreducible},$$

$$D^{(j_1)} \otimes D^{(j_2)} \simeq D^{(j_1+j_2)} \oplus \cdots \oplus D^{(|j_1-j_2|)}. \tag{3.70}$$

3.5 Irreducible tensors, Wigner–Eckart theorem

Tensor operators in QM: (recap)

Let $U(\vec{\theta}\,)$ be the rotation operator on some Hilbert space $\mathcal{H}$ of qm. states $|\psi\rangle$:

$$|\psi\rangle \underset{R}{\longrightarrow} |\psi'\rangle = U(\vec{\theta}\,)\,|\psi\rangle, \tag{3.71}$$

$$|\vec{x}\rangle \underset{R}{\longrightarrow} |\vec{x}\,'\rangle = U(\vec{\theta}\,)\,|\vec{x}\rangle = \underbrace{|R\vec{x}\rangle, \quad R = R(\vec{\theta}\,) = \text{rotation matrix}}_{\text{defines the geometrical meaning of } U(\vec{\theta}\,)}, \tag{3.72}$$

$$\begin{aligned}\Rightarrow\ \hat{\vec{x}}\,' &= U(\vec{\theta}\,)\,\hat{\vec{x}}\,U(\vec{\theta}\,)^\dagger = U(\vec{\theta}\,)\,\hat{\vec{x}}\,U(\vec{\theta}\,)^\dagger \underbrace{\int \mathrm{d}^3\vec{x}\,|\vec{x}\rangle\langle\vec{x}|}_{=\mathbb{1}} \\ &= \int \mathrm{d}^3\vec{x}\,U(\vec{\theta}\,)\,\hat{\vec{x}}\,|R^{-1}\vec{x}\rangle\langle\vec{x}| = \int \mathrm{d}^3\vec{x}\,U(\vec{\theta}\,)\,R^{-1}\,\vec{x}\,|R^{-1}\vec{x}\rangle\langle\vec{x}| \\ &= \int \mathrm{d}^3\vec{x}\,R^{-1}\,\vec{x}\,|\vec{x}\rangle\langle\vec{x}| = R^{-1}\,\hat{\vec{x}}\int \mathrm{d}^3\vec{x}\,|\vec{x}\rangle\langle\vec{x}| = R^{-1}\,\hat{\vec{x}}.\end{aligned} \tag{3.73}$$

Vector and (rank-n) tensor operators defined by analogous behaviour under rotations:

$$\hat{\vec{v}}\,' = U(\vec{\theta}\,)\,\hat{\vec{v}}\,U(\vec{\theta}\,)^\dagger = R^{-1}\,\hat{\vec{v}}, \tag{3.74}$$

$$T'_{a_1\ldots a_n} = U(\vec{\theta}\,)\,T_{a_1\ldots a_n}\,U(\vec{\theta}\,)^\dagger = \sum_{a'_1,\ldots,a'_n} (R^{-1})_{a_1a'_1}\cdots(R^{-1})_{a_na'_n}\,T_{a'_1\ldots a'_n}. \tag{3.75}$$

Infinitesimal rotations:

$$U(\delta\vec{\theta}\,) = \mathbb{1} - \mathrm{i}\delta\vec{\theta}\,\vec{J} + \ldots, \tag{3.76}$$

$$R(\delta\vec{\theta}\,) = \mathbb{1} - \mathrm{i}\delta\vec{\theta}\,\vec{J}^{(R)} + \ldots, \qquad (J_a^{(R)})_{bc} = -\mathrm{i}\epsilon_{abc}. \tag{3.77}$$

$\Rightarrow$ Transformation property (3.75) implies commutation relations: $(\hat{v}_a \equiv T_a)$

$$[J_a, T_{a_1\ldots a_n}] = \mathrm{i}\sum_{a'_1}\epsilon_{aa_1a'_1}T_{a'_1\ldots a_n} + \cdots + \mathrm{i}\sum_{a'_n}\epsilon_{aa_na'_n}T_{a_1\ldots a'_n}. \tag{3.78}$$

Note: Cartesian tensors $T_{a_1\ldots a_n}$ in general have the flaw of being reducible.

Example: rank-2 tensor T_{ab}.

$$T_{ab} = \underbrace{\tfrac{1}{3}\,\mathrm{Tr}(T)}_{\equiv S_0}\,\delta_{ab} + \underbrace{\tfrac{1}{2}(T_{ab} - T_{ba})}_{\equiv A_{ab}} + \underbrace{\left[\tfrac{1}{2}((T_{ab} + T_{ba}) - \tfrac{1}{3}\,\mathrm{Tr}(T)\,\delta_{ab}\right]}_{\equiv S_{ab}}. \tag{3.79}$$

The parts S_0, A_{ab}, S_{ab} transform independently:

- $S_0 = \mathrm{Tr}(T) = \sum_a T_{aa}$ = invariant, i.e. S_0 defines a "scalar".
- A_{ab} = antisymmetric, i.e. $A_a \equiv \sum_{c,b}\epsilon_{abc}A_{bc}$ defines a (pseudo)vector.
- S_{ab} = traceless symmetric = irreducible rank-2 part of T.

Irreducible (spherical) tensors:

$\hookrightarrow$ Definition via irreducible SU(2) representations $D^{(j)}$:

A set of $(2j+1)$ operators $T_m^{(j)}$ $(m=-j,-j+1,\dots,j)$ for a fixed $j=0,\frac{1}{2},1,\dots$ is called "irreducible (spherical) tensor operator" of rank j if it behaves as

$$T^{(j)\prime} = U(\vec{\theta}\,)\,T^{(j)}\,U(\vec{\theta}\,)^\dagger = D^{(j)}(\vec{\theta}\,)^{\mathrm{T}}\,T^{(j)}, \qquad T^{(j)} \equiv \begin{pmatrix} T_{+j}^{(j)} \\ \vdots \\ T_{-j}^{(j)} \end{pmatrix}. \tag{3.80}$$

$\hookrightarrow$ Irreducibility is implied by the irred. of $D^{(j)}$, i.e. all components $T_m^{(j)}$ can be obtained from a single component via symmetry relations (rotations).

Construction of spherical from cartesian tensors:

Recall spherical harmonics Y_{lm} (which transform like spherical tensors!):

$$Y_{lm}(\vartheta,\varphi) = \langle\vec{e}|l,m\rangle, \qquad \vec{e} = \text{unit vector with polar coordinates } \vartheta,\varphi \tag{3.81}$$

$$\begin{aligned} Y_{lm}(\vartheta',\varphi') &= \langle\vec{e}|\,U(\vec{\theta}\,)\,|l,m\rangle \qquad (\vartheta',\varphi' \text{ correspond to } \vec{e}\,'=R^{-1}\vec{e}.) \\ &= \sum_{m'}\langle\vec{e}|l,m'\rangle\,\langle l,m'|\,U(\vec{\theta}\,)\,|l,m\rangle, \qquad \sum_{m'}|l,m'\rangle\langle l,m'| = \mathbb{1}_{2l+1} \text{ on } D^{(l)} \\ &= \sum_{m'} Y_{lm'}(\vartheta,\varphi)\,D^{(l)}_{m'm}(\vec{\theta}\,) = \sum_{m'} D^{(l)}_{mm'}(\vec{\theta}\,)^{\mathrm{T}}\,Y_{lm'}(\vartheta,\varphi). \end{aligned} \tag{3.82}$$

Note: $r^l Y_{lm}(\vartheta,\varphi)$ = homogeneous polynomial of degree l in coordinates x_1,x_2,x_3, where $\vec{x}=r\vec{e}=(x_1,x_2,x_3)^{\mathrm{T}}$.

Procedure to construct $T_m^{(l)}$ out of some given $T_{a_1\dots a_l}$:

Calculate symmetrized version $\bar{T}_{a_1\dots a_l}$ of $T_{a_1\dots a_l}$ and define

$$T_m^{(l)} = \underbrace{\sqrt{\frac{4\pi}{2l+1}}}_{\text{or any other normalization}}\, r^l\,Y_{lm}(\vartheta,\varphi)\Big|_{x_{a_1}\cdots x_{a_l}\to\bar{T}_{a_1\dots a_l}}. \tag{3.83}$$

(Symmetrization of T necessary to obtain a unique correspondence!)

Proof of irreducibility:

$$\begin{aligned} T_m^{(l)\prime} &= U(\vec{\theta}\,)\,T_m^{(l)}\,U(\vec{\theta}\,)^\dagger = \sqrt{\frac{4\pi}{2l+1}}\,r^l\,Y_{lm}(\vartheta,\varphi)\Big|_{x_{a_1}\cdots x_{a_l}\to\bar{T}'_{a_1\dots a_l}=\sum_{a'_1,\dots,a'_l}(R^{-1})_{a_1a'_1}\cdots\bar{T}_{a_1\dots a_l}} \\ &= \sqrt{\frac{4\pi}{2l+1}}\,r^l\,Y_{lm}(\vartheta',\varphi')\Big|_{x_{a_1}\cdots x_{a_l}\to\bar{T}_{a_1\dots a_l}} \\ &= \sqrt{\frac{4\pi}{2l+1}}\,r^l\,\sum_{m'} D^{(l)}_{mm'}(\vec{\theta}\,)^{\mathrm{T}}\,Y_{lm'}(\vartheta,\varphi)\Big|_{x_{a_1}\cdots x_{a_l}\to\bar{T}_{a_1\dots a_l}} = \sum_{m'} D^{(l)}_{mm'}(\vec{\theta}\,)^{\mathrm{T}}\,T^{(l)}_{m'}. \qquad \# \end{aligned}$$

Examples:

- $l = 0:\quad T_0 = \text{scalar} \;\to\; T^{(0)}, \quad \sqrt{4\pi}\, r^0\, Y_{00} \equiv 1$, trivial case!
- $l = 1:\quad \vec{T} = (T_a) = \text{vector} \;\to\; T^{(1)}$.

$$\begin{aligned}
\sqrt{\frac{4\pi}{3}}\, r^1\, Y_{1,\pm1} &= \mp(x_1 \pm \mathrm{i}x_2)/\sqrt{2} &&\to\; \mp(T_1 \pm \mathrm{i}T_2)/\sqrt{2} \;\equiv\; T^{(1)}_{\pm1},\\
\sqrt{\frac{4\pi}{3}}\, r^1\, Y_{1,0} &= x_3 &&\to\; T_3 \;\equiv\; T^{(1)}_0.
\end{aligned} \tag{3.84}$$

- $l = 2:\quad T_{ab} = \text{rank-2 tensor} \;\to\; T^{(2)}$.

$$\begin{aligned}
\sqrt{\frac{4\pi}{5}}\, r^2\, Y_{2,\pm2} &= \sqrt{\frac{3}{8}}(x_1^2 - x_2^2 \pm 2\mathrm{i}x_1x_2) &&\to\; \sqrt{\frac{3}{8}}[T_{11} - T_{22} \pm \mathrm{i}(T_{12} + T_{21})] \;\equiv\; T^{(2)}_{\pm2},\\
\sqrt{\frac{4\pi}{5}}\, r^2\, Y_{2,\pm1} &= \mp\sqrt{\frac{3}{2}}(x_1 \pm \mathrm{i}x_2)x_3 &&\to\; \mp\sqrt{\frac{3}{8}}[T_{13} + T_{31} \pm \mathrm{i}(T_{23} + T_{32})] \;\equiv\; T^{(2)}_{\pm1},\\
\sqrt{\frac{4\pi}{5}}\, r^2\, Y_{2,0} &= \frac{1}{2}(2x_3^2 - x_1^2 - x_2^2) &&\to\; \frac{1}{2}(2T_{33} - T_{11} - T_{22}) \;\equiv\; T^{(2)}_0.
\end{aligned} \tag{3.85}$$

Commutator relations for $T^{(j)}$ from infinitesimal rotations:

$$\begin{aligned}
U(\delta\vec{\theta}) &= \mathbb{1} - \mathrm{i}\delta\vec{\theta}\,\vec{J} + \ldots,\\
D^{(j)}(\delta\vec{\theta}) &= \mathbb{1} - \mathrm{i}\delta\vec{\theta}\,\vec{J}^{(j)} + \ldots.
\end{aligned} \tag{3.86}$$

$$\Rightarrow\; [\vec{J}, T^{(j)}_m] = \sum_{m'} T^{(j)}_{m'}\, \underbrace{\vec{J}^{(j)}_{m'm}}_{=\,\langle j,m'|\,\vec{J}\,|j,m\rangle},$$

$$[J_3, T^{(j)}_m] = m\, T^{(j)}_m, \qquad [J_\pm, T^{(j)}_m] = \sqrt{j(j+1) - m(m\pm1)}\, T^{(j)}_{m\pm1}. \tag{3.87}$$

Compare with

$$J_3\,|j,m\rangle = m\,|j,m\rangle, \qquad J_\pm\,|j,m\rangle = \sqrt{j(j+1) - m(m\pm1)}\,|j,m\pm1\rangle. \tag{3.88}$$

$\Rightarrow\; T^{(j_1)}_{m_1}\,|j_2,m_2\rangle$ behaves under rotations like $|j_1,m_1\rangle\,|j_2,m_2\rangle$:

$$\begin{aligned}
\vec{J}\,T^{(j_1)}_{m_1}\,|j_2,m_2\rangle &= [\vec{J}, T^{(j_1)}_{m_1}]\,|j_2,m_2\rangle + T^{(j_1)}_{m_1}\,\vec{J}\,|j_2,m_2\rangle,\\
J_3\,T^{(j_1)}_{m_1}\,|j_2,m_2\rangle &= (m_1 + m_2)\,T^{(j_1)}_{m_1}\,|j_2,m_2\rangle,\\
J_\pm\,T^{(j_1)}_{m_1}\,|j_2,m_2\rangle &= \sqrt{j_1(j_1+1) - m_1(m_1\pm1)}\,T^{(j_1)}_{m_1\pm1}\,|j_2,m_2\rangle\\
&\quad + \sqrt{j_2(j_2+1) - m_2(m_2\pm1)}\,T^{(j_1)}_{m_1}\,|j_2,m_2\pm1\rangle.
\end{aligned} \tag{3.89}$$

Wigner–Eckart theorem

The matrix elements of an irreducible tensor operator $T_m^{(j)}$ between angular momentum eigenstates $|\alpha, j, m\rangle$ obey: ($\alpha^{(\prime)}$ = remaining quantum numbers)

$$\langle \alpha, j, m| \, T_{m_1}^{(j_1)} \, |\alpha', j_2, m_2\rangle = \underbrace{\langle j, m|j_1, j_2; m_1, m_2\rangle}_{\text{CG coefficient}} \cdot \frac{\langle \alpha, j|| \, T^{(j_1)} \, ||\alpha', j_2\rangle}{\sqrt{2j+1}}, \tag{3.90}$$

$\langle \dots ||T^{(j_1)}|| \dots \rangle$ = "reduced matrix element", independent of m, m_1, m_2.

Proof based on the analogy between $T_{m_1}^{(j_1)} \, |j_2, m_2\rangle$ and $|j_1, m_1\rangle \, |j_2, m_2\rangle$:

⇒ Modify recursive calculation of CG coefficients described in Section 3.4:

- Procedure for each j-value:

  Construct $\{|j, m\rangle\}_{m=j,j-1,\dots,-j}$ for $j = j_1 + j_2$, then $j = j_1 + j_2 - 1, \dots, j = |j_1 - j_2|$.

  Previously: $|j, m\rangle$ expressed in terms of $|j_1, m_1\rangle \, |j_2, m_2\rangle$.

  Now: $|j, m\rangle$ expressed in terms of $T_{m_1}^{(j_1)} \, |j_2, m_2\rangle$.

- Highest m-values for fixed j:

  Previously: $|j, m = j\rangle$ fixed up to phase choice in terms of $|j_1, m_1\rangle \, |j_2, m_2\rangle$, e.g.
  $|j_1 + j_2, j_1 + j_2\rangle \equiv |j_1, j_2; m_1 = j_1, m_2 = j_2\rangle$,
  $|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle \perp$ known $|j_1 + j_2, j_1 + j_2 - 1\rangle$, etc.

  Now: $|j, m = j\rangle$ fixed by $T_{m_1=j_1}^{(j_1)} \, |j_2, m_2 = j_2\rangle$ up to some constant $A(j)$, since there is no canonical normalization of $T_{m_1}^{(j_1)} \, |j_2, m_2\rangle$ (in contrast to $|j_1, m_1\rangle \, |j_2, m_2\rangle$).

- Lower m-values for fixed j:

  Previously: Evaluate $J_-^{j-m} \, |j, j\rangle$ to derive relation:
  $|j, m\rangle = \sum_{m_1,m_2} |j_1, j_2; m_1, m_2\rangle \, \underbrace{\langle j_1, j_2; m_1, m_2|j, m\rangle}_{\text{explicitly constructed}}$.

  Now: The same procedure applied to $J_-^{j-m} \, |j, j\rangle \cdot A(j)$ yields

$$A(j) \, |j, m\rangle = \sum_{m_1,m_2} T_{m_1}^{(j_1)} \, |j_2, m_2\rangle \, \langle j_1, j_2; m_1, m_2|j, m\rangle. \tag{3.91}$$

- Solve (3.91) for $\langle j, m'| \, T_{m_1}^{(j_1)} \, |j_2, m_2\rangle$ upon evaluating $\langle j, m'| \cdot$ (3.91):

$$A(j) \, \delta_{mm'} = \sum_{m_1,m_2} \langle j, m'| \, T_{m_1}^{(j_1)} \, |j_2, m_2\rangle \, \langle j_1, j_2; m_1, m_2|j, m\rangle,$$

  and calculating $\sum_m \langle j, m|j_1, j_2; m_1', m_2'\rangle \cdots$:

$$A(j) \, \langle j, m'|j_1, j_2; m_1', m_2'\rangle = \sum_{m_1,m_2} \langle j, m'| \, T_{m_1}^{(j_1)} \, |j_2, m_2\rangle \times \underbrace{\sum_m \langle j_1, j_2; m_1, m_2|j, m\rangle \, \langle j, m|j_1, j_2; m_1', m_2'\rangle}_{=\delta_{m_1 m_1'} \, \delta_{m_2 m_2'}}$$

$$= \langle j, m'| \, T_{m_1'}^{(j_1)} \, |j_2, m_2'\rangle.$$

  ⇒ WE theorem ($A(j) \rightarrow$ reduced matrix element; α, α' suppressed in notation). #

Implications of the WE theorem:

- Qm. transition probabilities from some state $|j_2, m_2\rangle \to |j, m\rangle$ typically ruled by matrix elements such as

$$\langle j, m| \underbrace{T^{(j_1)}_{m_1}}_{\substack{\text{operator for interaction}\\ \text{driving the transition}}} |j_2, m_2\rangle \;=\; 0 \quad \text{if } \underbrace{m \neq m_1 + m_2 \;\text{ or }\; j \neq j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|}_{\text{selection rules implied by the WE theorem}}. \tag{3.92}$$

  E.g. $T^{(j_1)}$ = scalar $T^{(0)}$: only $j = j_2$ “allowed”,
  $T^{(j_1)}$ = vector $T^{(1)}$: only $j = j_2, j_2 \pm 1$.

- Relative strengths of transition matrix elements entirely given by CG coefficients:

$$\left| \frac{\langle j, m| T^{(j_1)}_{m_1} |j_2, m_2\rangle}{\langle j, m'| T^{(j_1)}_{m_1'} |j_2, m_2'\rangle} \right| = \left| \frac{\langle j, m|j_1, j_2; m_1, m_2\rangle}{\langle j, m'|j_1, j_2; m_1', m_2'\rangle} \right|. \tag{3.93}$$

3.6 Tensors of SO(N)

Definition: SO(N), $N \in \mathbb{N}$, is the group of real orthogonal $N \times N$ matrices R, $R^{\mathrm{T}} R = \mathbb{1}_N$, with $\det R = 1$ ("defining representation").

The matrices R form an N-dimensional (irreducible for $N > 2$) representation on the vector space $V = \mathbb{R}^N$:

$$v \in V : v^i \to v'^i = R^{ij} v^j. \tag{3.94}$$

A tensor $T^{i_1 \dots i_r}$ of rank r transforms like the tensor product of r vectors:

$$T^{i_1 \dots i_r} \to T'^{i_1 \dots i_r} = R^{i_1 j_1} \dots R^{i_r j_r} T^{j_1 \dots j_r}. \tag{3.95}$$

Properties:

- The tensor product of two tensors of ranks r_1 and r_2,
$$T_3^{i_1 \dots i_{r_1+r_2}} = T_1^{i_1 \dots i_{r_1}} T_2^{i_{r_1+1} \dots i_{r_1+r_2}}, \tag{3.96}$$
transforms as a tensor of rank $r_1 + r_2$.
- The contraction $\sum_j T^{i_1 \dots j \dots j \dots i_r}$ of a rank-r tensor transforms as a tensor of rank $r - 2$.
- The components of $T^{i_1 \dots i_r}$ furnish an N^r-dimensional representation D of SO(N):
$$\vec{T} = (T^{1 \dots 11}, T^{1 \dots 12}, \dots, T^{N \dots NN})^{\mathrm{T}} : \quad \vec{T}^a \to \vec{T}'^a = D^{ab} \vec{T}^b, \quad a, b = 1, \dots, N^r. \tag{3.97}$$

"Invariant symbols" are tensors that are invariant under group transformations (in a more general context "relative tensors", i.e. they receive a factor $(\det R)^w$ with some "weight" w when transformed by R). Invariant symbols follow from the defining properties of R:

- $RR^{\mathrm{T}} = \mathbb{1} \quad \Rightarrow \quad (\delta')^{ij} = R^{ik} R^{jl} \delta^{kl} = R^{ik} R^{jk} = R^{ik} (R^{\mathrm{T}})^{kj} = \delta^{ij}$,
- $1 = \det R = R^{1 i_1} \dots R^{N i_N} \epsilon^{i_1 \dots i_N} \quad \Rightarrow \quad (\epsilon')^{i_1 \dots i_N} \equiv R^{i_1 j_1} \dots R^{i_N j_N} \epsilon^{j_1 \dots j_N} = \epsilon^{i_1 \dots i_N}$.

Example: Reducibility of rank-2 tensors

The representations under which tensors of rank $r > 1$ transform are reducible. A rank-2 tensor T^{ij} can be decomposed according to

$$
\begin{aligned}
T^{ij} &= S^{ij} + A^{ij} + \frac{1}{N} \delta^{ij} S_0 \qquad \text{with} && \qquad (3.98)\\
S^{ij} &= \frac{1}{2} \left(T^{ij} + T^{ji} \right) - \frac{1}{N} \delta^{ij} S_0 && \text{symmetric and traceless,}\\
A^{ij} &= \frac{1}{2} (T^{ij} - T^{ji}) && \text{antisymmetric,}\\
S_0 &= T^{ii} && \text{scalar.}
\end{aligned}
$$

The S^{ij}, A^{ij}, and S_0 parts span invariant subspaces under group transformations: $T^{ij} \pm T^{ji} \to R^{ik} R^{jl} (T^{kl} \pm T^{lk})$. The representation decomposes as

$$\underbrace{N \otimes N}_{\text{general rank 2}} = \underbrace{\left(\frac{1}{2}N(N+1) - 1\right)}_{\text{sym. traceless}} \oplus \underbrace{\frac{1}{2}N(N-1)}_{\text{antisym.}} \oplus \underbrace{1}_{\text{trace}} . \tag{3.99}$$

For higher ranks, the symmetry patterns become more complicated. A full classification is possible in the formalism of "Young tableaux" which are related to the representations of the symmetric groups S_r (see, e.g., Chapter 5 in [9]).

Dual, self-dual, and anti-self-dual tensors

For a totally antisymmetric tensor $A^{i_1 \cdots i_r}$, its dual tensor $\tilde{A}^{i_1 \cdots i_{N-r}}$ is defined as

$$\tilde{A}^{i_1 \dots i_{N-r}} = \frac{1}{r!} \epsilon^{i_1 \dots i_N} A^{i_{N-r+1} \dots i_N} \tag{3.100}$$

and antisymmetric by construction. For SO($2N$), we can define the self-dual (+) and anti-self-dual (−) tensors

$$T_\pm^{i_1 \dots i_N} = \frac{1}{2}\left(A^{i_1 \dots i_N} \pm \tilde{A}^{i_1 \dots i_N}\right) \quad \Rightarrow \quad \tilde{T}_\pm^{i_1 \dots i_N} = \pm T_\pm^{i_1 \dots i_N} . \tag{3.101}$$

The self-dual and anti-self-dual tensors span invariant subspaces under group transformations.

Examples

- Special case SO(4): For $N = 4$, the 6-dimensional representation furnished by an antisymmetric tensor A^{ij} reduces to two 3-dimensional representations:

$$\underbrace{4 \otimes 4}_{\substack{\text{general} \\ \text{rank 2}}} = \underbrace{9}_{\substack{\text{sym.} \\ \text{trace-} \\ \text{less}}} \oplus \underbrace{3}_{\substack{\text{self-} \\ \text{dual}}} \oplus \underbrace{3}_{\substack{\text{anti-} \\ \text{self-} \\ \text{dual}}} \oplus \underbrace{1}_{\text{trace}} . \tag{3.102}$$

  This happens in a similar way (up to factors of i) in the Lorentz group SO(3, 1): Electromagnetic field strength tensor $F^{\mu\nu}$ and its dual $\tilde{F}^{\mu\nu} \to F_\pm^{\mu\nu} = F^{\mu\nu} \pm \mathrm{i}\tilde{F}^{\mu\nu}$.

- Special case SO(3):

$$A^{ij} = \begin{pmatrix} 0 & A^3 & -A^2 \\ -A^3 & 0 & A^1 \\ A^2 & -A^1 & 0 \end{pmatrix} \quad \to \quad \frac{1}{2}\epsilon^{kij} A^{ij} = \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix} \tag{3.103}$$

  ⇒ It is always possible to trade a pair of antisymmetric indices for one index.
  ⇒ It is sufficient to regard symmetric traceless tensors when studying irreducible representations of SO(3). Number of components:

Symmetric tensor of rank r: $\displaystyle\sum_{n_1=0}^{r}\sum_{n_2=0}^{r-n_1} 1 = \frac{1}{2}(r+1)(r+2)$ components
(n_1 indices have the value 1, n_2 the value 2, $n_3 = r - n_1 - n_2$ the value 3).
Each pair of indices can be contracted. $\Rightarrow$ $\frac{1}{2}r(r-1)$ trace conditions.
Traceless symmetric tensor: $\displaystyle\frac{1}{2}(r+1)(r+2) - \frac{1}{2}r(r-1) = 2r+1$ components
($\widehat{=}\ 2l+1$ components of a spherical tensor $T^{(l)}$).

The Lie algebra so(N)

As shown in Section 3.1, with the convention that SO(N) elements are expressed as $R = \exp\{-\mathrm{i}\theta_a J_a\}$, the generators J_a of SO(N) are hermitian and antisymmetric (i.e. $\mathrm{i}J_a$ is real and antisymmetric).
$\Rightarrow$ There are $\frac{1}{2}N(N-1)$ generators. In the defining representation, the generators can be chosen as

$$J^{ij}_{(mn)} = \mathrm{i}\big(\delta^{mj}\delta^{ni} - \delta^{mi}\delta^{nj}\big), \tag{3.104}$$

where (mn), $m > n$, takes the values $(mn) \equiv a = 1, \ldots, \frac{1}{2}N(N-1)$, and $J_{(nm)} = -J_{(mn)}$.
Lie algebra so(N) (independent of the representation!):

$$[J_{(mn)}, J_{(pq)}] = \mathrm{i}\big(\delta^{mp}J_{(nq)} + \delta^{nq}J_{(mp)} - \delta^{mq}J_{(np)} - \delta^{np}J_{(mq)}\big) \equiv \mathrm{i}f_{(mn)(pq)c}J_c, \tag{3.105}$$

where the last equality defines the structure constants f_{abc}.
Every antisymmetric tensor A^{ij} can be expressed as $A^{ij} = \mathrm{i}\mathcal{A}_a J^{ij}_a$, $\mathcal{A}_a \in \mathbb{R}$, i.e. in a basis J_a of generators it can be represented by the coefficients $\mathcal{A}_a$.
$\hookrightarrow$ How do the $\mathcal{A}_a$ transform under an SO(N) transformation with group parameters θ_a?

$$A'^{ij} = R^{ik}(\theta)R(\theta)^{jl}A^{kl} = R(\theta)^{ik}A^{kl}(R(\theta)^{-1})^{lj} \quad \Rightarrow \quad A' = R(\theta)AR(\theta)^{-1}. \tag{3.106}$$

Transformation with infinitesimal θ_a:

$$\begin{aligned}\delta A = A' - A &= (\mathbb{1} - \mathrm{i}\theta_a J_a)A(\mathbb{1} + \mathrm{i}\theta_b J_b) - A = -\mathrm{i}\theta_a[J_a, A] = \theta_a\mathcal{A}_b[J_a, J_b]\\ &= \mathrm{i}\theta_a\mathcal{A}_b f_{abc}J_c.\end{aligned} \tag{3.107}$$

On the other hand, with $A' = \mathrm{i}\mathcal{A}'_a J_a$ and $\mathcal{A}'_a = \mathcal{A}_a + \delta\mathcal{A}_a$,

$$\begin{aligned}\delta A &= \mathrm{i}\mathcal{A}'_c J_c - \mathrm{i}\mathcal{A}_c J_c = \mathrm{i}\delta\mathcal{A}_c J_c\\ \Rightarrow \quad \mathcal{A}'_c &= \big(\delta_{cb} + \theta_a f_{abc}\big)\mathcal{A}_b. \equiv \big(\delta_{cb} - \mathrm{i}\theta_a(F_a)_{cb}\big)\mathcal{A}_b.\end{aligned} \tag{3.108}$$

$\Rightarrow$ $\mathcal{A}_a$ transforms under the adjoint representation with the generators

$$(F_a)_{bc} = \mathrm{i}f_{acb} = -\mathrm{i}f_{abc}. \tag{3.109}$$

Example: so(4)

SO(4) has six generators:

$$J_{(12)} \equiv J_3, \quad J_{(23)} \equiv J_1, \quad J_{(31)} \equiv J_2, \quad J_{(14)} \equiv K_1, \quad J_{(24)} \equiv K_2, \quad J_{(34)} \equiv K_3.$$

The Lie algebra is (verify this!)

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k, \qquad [J_i, K_j] = \mathrm{i}\epsilon_{ijk} K_k, \qquad [K_i, K_j] = \mathrm{i}\epsilon_{ijk} J_k. \tag{3.110}$$

$\hookrightarrow$ J_i, $i = 1, 2, 3$, generate the SO(3) rotations in the x_1-x_2-x_3 space.
$\hookrightarrow$ K_i, $i = 1, 2, 3$, transform like the components of a vector $\vec{K} \in$ SO(3).

Choose a new basis $T_{1,i} = \frac{1}{2}(J_i + K_i)$, $T_{2,i} = \frac{1}{2}(J_i - K_i)$. Lie algebra in this basis:

$$[T_{1,i}, T_{1,j}] = \mathrm{i}\epsilon_{ijk} T_{1,k}, \qquad [T_{2,i}, T_{2,j}] = \mathrm{i}\epsilon_{ijk} T_{2,k}, \qquad [T_{1,i}, T_{2,j}] = 0. \tag{3.111}$$

$\Rightarrow$ The Lie algebra so(4) falls apart into two su(2) algebras, so(4) $\simeq$ su(2) $\times$ su(2).
$\Rightarrow$ The group SO(4) is locally isomorphic to SU(2) $\times$ SU(2)
(SU(2) $\times$ SU(2) is a universal cover of SO(4)).

3.7 Tensors of SU(N)

Definition: SU(N), $N \in \mathbb{N}$: the group of unitary $N \times N$ matrices U, $U^\dagger U = \mathbb{1}_N$, with $\det U = 1$ ("defining representation").

The matrices U form an N-dimensional (irreducible for $N > 1$) representation on the vector space $V = \mathbb{C}^N$:

$$u \in V : u^i \to u'^i = U^i{}_j u^j. \tag{3.112}$$

The transformations U leave the scalar product $v^\dagger u$ invariant:

$$v^\dagger u = v^\dagger U^\dagger U u \quad \Leftrightarrow \quad (v^i)^* u^i = (v^i)^* (U^j{}_i)^* U^j{}_k u^k. \tag{3.113}$$

$\Rightarrow$ v^* transforms with the complex conjugate representation U^*: $(v^*)^i \to (U^*)^i{}_j (v^*)^j$.
$\hookrightarrow$ Define $v_i \equiv (v^*)^i$ with a lower index. Lower indices transform with U^*, while upper indices transform with U. We can then write

$$v'_i u'^i = \left((U^*)^i{}_j (v^*)^j\right)\left(U^i{}_k u^k\right) = v_j (U^\dagger)^j{}_i U^i{}_k u^k = v_i u^i, \tag{3.114}$$

where contractions are always performed between upper and lower indices (sometimes the notation $U_i{}^j \equiv (U^\dagger)^j{}_i$ is used so that $v'_i = U_i{}^j v_j$). Contractions $v^i u^i$ and $v_i u_i$ do not transform as scalars and are (in this sense) not defined.

Tensors of SU(N) can carry both upper and lower indices and transform as

$$T^{i_1 \ldots i_n}_{j_1 \ldots j_m} \to T'^{i_1 \ldots i_n}_{j_1 \ldots j_m} = U^{i_1}{}_{k_1} \ldots U^{i_n}{}_{k_n} T^{k_1 \ldots k_n}_{l_1 \ldots l_m} (U^\dagger)^{l_1}{}_{j_1} \ldots (U^\dagger)^{l_m}{}_{j_m}. \tag{3.115}$$

Invariant symbols:

- $(U^\dagger)^i{}_j U^j{}_k = \delta^i_k \quad \Rightarrow \quad \delta^i_k \to \delta'^i_k = (U^\dagger)^i{}_j \delta^j_l U^l{}_k = \delta^i_k$.
  There are no invariant symbols δ^{ij} and δ_{ij} ⇒ Traces wrt. two upper (rsp. two lower) indices do not transform as tensors.
- $\det U = 1 \quad \Rightarrow \quad \epsilon^{i_1\ldots i_N} \to \epsilon'^{i_1\ldots i_N} = U^{i_1}{}_{j_1} \ldots U^{i_N}{}_{j_N} \epsilon^{j_1\ldots j_N} = \epsilon^{i_1\ldots i_N}$.
- $\det U^\dagger = 1 \quad \Rightarrow \quad \epsilon_{i_1\ldots i_N} \to \epsilon'_{i_1\ldots i_N} = \epsilon_{j_1\ldots j_N} (U^\dagger)^{j_1}{}_{i_1} \ldots (U^\dagger)^{j_N}{}_{i_N} = \epsilon_{i_1\ldots i_N}$.

Special case SU(2):

- For $N = 2$, $U(\vec\phi) = \exp\{-\mathrm{i}\vec\phi \cdot \vec\sigma/2\}$ and $U^*(\vec\phi) = \exp\{\mathrm{i}\vec\phi \cdot \vec\sigma^*/2\}$ are equivalent. For infinitesimal $\vec\phi$:
  $$U(\vec\phi)^i{}_j = \delta^i_j - \frac{\mathrm{i}}{2}\phi_a(\sigma_a)^i{}_j, \qquad U^*(\vec\phi)_i{}^j = \delta^j_i + \frac{\mathrm{i}}{2}\phi_a(\sigma^*_a)_i{}^j = \epsilon_{ik} U(\vec\phi)^k{}_l \epsilon^{lj}, \tag{3.116}$$
  because $\qquad \epsilon_{ik}(\sigma_a)^k{}_l \epsilon^{lj} = -(\sigma^*_a)_i{}^j$.

  ⇒ SU(2) is pseudoreal and has the antisymmetric invariant bilinear form $v^{\mathrm{T}}\epsilon u = v^j \epsilon_{ij} u^i$, $\epsilon^{\mathrm{T}} = -\epsilon$.
- A tensor with n upper and m lower indices can always be expressed as an equivalent tensor with $n + m$ upper (or lower) indices:
  $$T^{i_1\ldots i_n}_{j_1\ldots j_m} \quad \to \quad T^{i_1\ldots i_n j_1\ldots j_m} = T^{i_1\ldots i_n}_{k_1\ldots k_m} \epsilon^{j_1 k_1} \ldots \epsilon^{j_m k_m}. \tag{3.117}$$
- Antisymmetric contributions in any two indices span invariant subspaces:
  $\epsilon_{jk} T^{i_1\ldots j\ldots k\ldots i_r}$ transforms as a rank $r-2$ tensor.
- Number of independent components $T^{1\ldots1}$, $T^{1\ldots12}$, …, $T^{1\ldots12\ldots2}$, …, $T^{2\ldots2}$ of a symmetric tensor $T^{i_1\ldots i_r}$: $r+1$.

Special case SU(3):

- Similarly to SO(3), ϵ^{ijk} can be used to trade two antisymmetric lower indices for one upper index (analogously for ϵ_{ijk}), i.e. antisymmetric contributions can be expressed as symmetric tensors of lower rank.
  ⇒ Tensors that are totally symmetric in all upper indices and in all lower indices always span invariant subspaces.
- The trace $\delta^{j_1}_{i_1} T^{i_1\ldots i_n}_{j_1\ldots j_m}$ (symmetry ⇒ all traces are equivalent) spans an invariant subspace.
- Number of components of a traceless tensor $T^{i_1\ldots i_n}_{j_1\ldots j_m}$ with all upper and all lower indices symmetric:
  $$\underbrace{\frac{1}{2}(n+1)(n+2)}_{n \text{ sym. upper ind.}} \cdot \underbrace{\frac{1}{2}(m+1)(m+2)}_{m \text{ sym. lower ind.}} - \underbrace{\frac{1}{2}n(n+1) \cdot \frac{1}{2}m(m+1)}_{\text{trace, rank } (n-1,m-1) \text{ sym. tensor}}$$
  $$= \frac{1}{2}(n+1)(m+1)(n+m+2). \tag{3.118}$$

Dimensions of the irreducible representations (n, m) of SU(3) up to $m = n = 3$:

(n,m)	$n=0$	1	2	3
$m=0$	1	3	6	10
1	3^*	8	15	24
2	6^*	15^*	27	42
3	10^*	24^*	42^*	64

Besides (n, m) the dimension can be used to label irreducible representations. Representations with $n < m$ are then labelled by $\dim(n, m)^*$ to distinguish them from (m, n), e.g. $(1, 0) \equiv 3$, $(0, 1) \equiv 3^*$; $(m, n) \simeq (n, m)^*$.

- **Clebsch-Gordan series for SU(3)**

Given two irreducible tensors $A^{\{i_1\ldots i_n\}}_{\{j_1\ldots j_m\}}$ and $B^{\{i_1\ldots i_{n'}\}}_{\{j_1\ldots j_{m'}\}}$ ($\{\ldots\}$ means that the indices are totally symmetric). How does the tensor product $T^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}} = A^{\{i_1\ldots i_n\}}_{\{j_1\ldots j_m\}} B^{\{i_1\ldots i_{n'}\}}_{\{j_1\ldots j_{m'}\}}$ decompose into irreducible representations?

1. Recursively take out all traces:

$$\delta^{l_1}_{i_1} T^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}}, \qquad \delta^{k_1}_{j_1} T^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}},$$

$$\delta^{l_1}_{i_1}\delta^{l_2}_{i_2} T^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}}, \qquad \delta^{l_1}_{i_1}\delta^{k_1}_{j_1} T^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}}, \qquad \delta^{k_1}_{j_1}\delta^{k_2}_{j_2} T^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}},$$

$$\ldots$$

$\hookrightarrow$ Produces a traceless tensor $\tilde{T}^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}}$ that transforms under a (reducible, because $\tilde{T}$ is not yet totally symmetric) representation labelled by $(n, m; n', m')$.

$$\Rightarrow \quad (n,m) \otimes (n',m') = \bigoplus_{p=0}^{\min(n,m')} \bigoplus_{q=0}^{\min(n',m)} (n-p, m-q; n'-q, m'-p). \tag{3.119}$$

2. Recursively take out antisymmetric contributions from traceless tensors:

$$\epsilon_{i_1 j_1 s_1} \tilde{T}^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}}, \qquad \epsilon^{k_1 l_1 t_1} \tilde{T}^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}},$$

$$\epsilon_{i_1 j_1 s_1}\epsilon_{i_2 j_2 s_2} \tilde{T}^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}}, \qquad \epsilon^{k_1 l_1 t_1}\epsilon^{k_2 l_2 t_2} \tilde{T}^{\{i_1\ldots i_n\}\{j_1\ldots j_{n'}\}}_{\{k_1\ldots k_m\}\{l_1\ldots l_{m'}\}},$$

$$\ldots$$

Note that e.g. contraction with $\epsilon_{i_1 j_1 s_1}$ automatically results in symmetric lower indices (verify this!). Analogously for, e.g., $\epsilon^{k_1 l_1 t_1}$.

$$\Rightarrow \quad (n,m;n',m') = (n+n', m+m') \oplus \bigoplus_{p=1}^{\min(n,n')} (n+n'-2p, m+m'+p) \oplus \bigoplus_{p=1}^{\min(m,m')} (n+n'+p, m+m'-2p). \tag{3.120}$$

Example:

$$
\begin{aligned}
(1,1)\otimes(1,1) &= (1,1;1,1)\oplus(1,0;0,1)\oplus(0,1;1,0)\oplus(0,0;0,0) \\
\text{with}\quad (1,1;1,1) &= (2,2)\otimes(3,0)\otimes(0,3), \\
(1,0;0,1) &= (1,1), \\
(0,1;1,0) &= (1,1), \\
(0,0;0,0) &= (0,0).
\end{aligned}
$$

$$
\begin{array}{lccccccccccccc}
\Rightarrow & (1,1) & \otimes & (1,1) & = & (2,2) & \oplus & (3,0) & \oplus & (0,3) & \oplus & (1,1) & \oplus & (1,1) \oplus (0,0), \\
\Leftrightarrow & 8 & \otimes & 8 & = & 27 & \oplus & 10 & \oplus & 10^{*} & \oplus & 8 & \oplus & 8 \oplus 1.
\end{array}
$$

Chapter 4

SU(3)

4.1 The su(3) algebra, roots, and weights

The defining representation of the algebra su(3) consists of traceless hermitian matrices. A common basis choice is given by the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

which generalise the Pauli matrices from su(2) (it is straightforward to write down a basis for su(N) for any N).

↪ SU(3) generators in the fundamental representation: $T_a = \frac{1}{2}\lambda_a$, $a = 1, \ldots, 8$.
↪ Normalisation: $\mathrm{Tr}\, T^a T^b = T_F \delta^{ab}$, $T_F = \frac{1}{2}$.
↪ Lie algebra $[T^a, T^b] = \mathrm{i} f^{abc} T^c$, f^{abc} totally antisymmetric with non-zero components

$$\begin{aligned} f^{123} &= 1, & f^{147} &= f^{246} = f^{257} = f^{345} = \frac{1}{2}, \\ f^{156} &= f^{367} = -\frac{1}{2}, & f^{458} &= f^{678} = \frac{\sqrt{3}}{2}. \end{aligned} \tag{4.1}$$

In the fundamental representation, the anti-commutator has the form

$$\{T_a, T_b\} = \frac{1}{3}\,\delta_{ab} + d_{abc} T_c \qquad \Rightarrow \qquad T_a T_b = \frac{1}{6}\,\delta_{ab} + \frac{1}{2}(d_{abc} + \mathrm{i} f_{abc}) T_c, \tag{4.2}$$

where d_{abc} is totally symmetric with non-zero components

$$\begin{aligned} d_{118} = d_{228} = d_{338} = -d_{888} &= \frac{1}{\sqrt{3}}, \\ d_{448} = d_{558} = d_{668} = d_{778} &= -\frac{1}{2\sqrt{3}}, \\ d_{146} = d_{157} = d_{256} = d_{344} = d_{355} = -d_{247} = -d_{366} = -d_{377} &= \frac{1}{2}. \end{aligned} \tag{4.3}$$

su(3) contains three "overlapping" su(2) subalgebras. Defining

$$I_{1,2,3} = T^{1,2,3}, \qquad U_{1,2} = T^{6,7}, \qquad V_{1,2} = T^{4,5}, \qquad Y = \frac{2}{\sqrt{3}}T^8, \tag{4.4}$$

- $[I_1, I_2] = \mathrm{i}I_3$ (cyclic),
- $[U_1, U_2] = \mathrm{i}\frac{1}{2}\left(I_3 + \frac{3}{2}Y\right) \equiv \mathrm{i}U_3$ (cyclic),
- $[V_1, V_2] = \mathrm{i}\frac{1}{2}\left(-I_3 + \frac{3}{2}Y\right) \equiv \mathrm{i}V_3$ (cyclic).

I_3, U_3, V_3 are not independent $\Rightarrow$ su(3) $\not\simeq$ su(2) $\times$ su(2) $\times$ su(2).

Definition: The number of simultaneously diagonalisable generators is called the *rank* of the Lie algebra.

su(3) has rank 2; choose I_3 and Y which are already diagonal.
$\Rightarrow$ Classify states by their eigenvalues of I_3 and Y:

$$I_3|i_3, y\rangle = i_3|i_3, y\rangle, \qquad Y|i_3, y\rangle = y|i_3, y\rangle. \tag{4.5}$$

Definition: The vectors $\vec{\omega} = (i_3, y)$ of eigenvalues of the diagonal generators are called *weights* of the *weight vectors* $|\vec{\omega}\rangle \equiv |i_3, y\rangle$.

Definition: The non-zero vectors $\vec{\alpha} = (\Delta i_3, \Delta y)$ for which there exists an $X_\alpha \in \mathrm{su}(3)_{\mathbb{C}}$ [complexification of su(3): all linear combinations of T^a with complex coefficients; $\mathrm{su}(3)_{\mathbb{C}} \simeq \mathrm{sl}(3, \mathbb{C})$], so that

$$[\vec{H}, X_\alpha] = \vec{\alpha} X_\alpha \qquad \text{with} \qquad \vec{H} = (I_3, Y), \tag{4.6}$$

are called the *roots* of su(3). X_α is called the *root vector* corresponding to the root $\vec{\alpha}$. In other words, X_α is a common eigenvector of ad_{I_3} and ad_Y with eigenvalues Δi_3 and Δy.

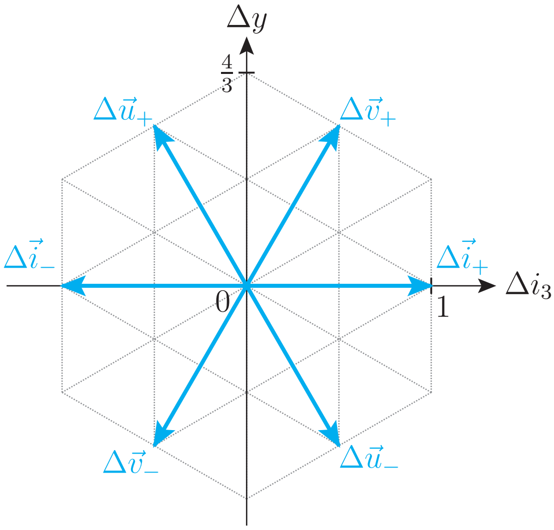
su(3) has six root vectors $I_\pm$, $U_\pm$, $V_\pm$ with roots $\Delta\vec{i}_\pm$, $\Delta\vec{u}_\pm$, $\Delta\vec{v}_\pm$:

$$\begin{aligned} I_\pm = I_1 \pm \mathrm{i}I_2 : &\quad [I_3, I_\pm] = \pm I_\pm, &\quad [Y, I_\pm] = 0 &\quad \Rightarrow \quad \Delta\vec{i}_\pm = (\pm 1, 0), \\ U_\pm = U_1 \pm \mathrm{i}U_2 : &\quad [I_3, U_\pm] = \mp\frac{1}{2}U_\pm, &\quad [Y, U_\pm] = \pm U_\pm &\quad \Rightarrow \quad \Delta\vec{u}_\pm = (\mp\tfrac{1}{2}, \pm 1), \\ V_\pm = V_1 \pm \mathrm{i}V_2 : &\quad [I_3, V_\pm] = \pm\frac{1}{2}V_\pm &\quad [Y, V_\pm] = \pm V_\pm &\quad \Rightarrow \quad \Delta\vec{v}_\pm = (\pm\tfrac{1}{2}, \pm 1). \end{aligned} \tag{4.7}$$

In the basis $I_\pm$, $U_\pm$, $V_\pm$, I_3, Y, the commutators not listed in (4.7) are

$$\begin{aligned}
&[I_+, I_-] = 2I_3, && [I_+, U_+] = V_+, && [I_+, U_-] = 0, \\
&[U_+, U_-] = -I_3 + \frac{3}{2}Y, && [I_+, V_-] = -U_-, && [I_+, V_+] = 0, \\
&[V_+, V_-] = I_3 + \frac{3}{2}Y, && [U_+, V_-] = I_-, && [U_+, V_+] = 0.
\end{aligned} \tag{4.8}$$

(remaining commutators by hermitian conjugation, e.g. $[I_-, U_-] = [I_+, U_+]^\dagger$).
Root diagram:



Of the six roots, only two are linearly independent.

- *Positive roots:* all roots in some given half-space. Common choice: $\Delta\vec{i}_+$, $\Delta\vec{u}_+$, $\Delta\vec{v}_+$.
- *Simple roots:* minimal subset of positive roots so that all positive roots can be expressed as linear combinations of simple roots with positive coefficients.
  Here: $\Delta\vec{v}_+ = \Delta\vec{i}_+ + \Delta\vec{u}_+ \quad\Rightarrow\quad \Delta\vec{i}_+$ and $\Delta\vec{u}_+$ are simple.

Applying a root vector X_α to a weight vector $|\vec{\omega}\rangle$ shifts the weight by $\vec{\alpha}$:

$$\begin{aligned}
\vec{H}X_\alpha|\vec{\omega}\rangle = \big(X_\alpha\vec{H} + [\vec{H}, X_\alpha]\big)|\vec{\omega}\rangle = \big(X_\alpha\vec{\omega} + \vec{\alpha}X_\alpha\big)|\vec{\omega}\rangle = (\vec{\omega} + \vec{\alpha})X_\alpha|\vec{\omega}\rangle \\
\Rightarrow \quad X_\alpha|\vec{\omega}\rangle \propto |\vec{\omega} + \vec{\alpha}\rangle
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\Rightarrow \quad I_\pm|i_3, y\rangle &\propto |i_3 \pm 1, y\rangle, \\
U_\pm|i_3, y\rangle &\propto |i_3 \mp \tfrac{1}{2}, y \pm 1\rangle, \\
V_\pm|i_3, y\rangle &\propto |i_3 \pm \tfrac{1}{2}, y \pm 1\rangle.
\end{aligned}$$

The proportionality constants may vanish for certain weights.

4.2 Irreducible representations

Possible values of i_3 and y:

- I_1, I_2, I_3 span an su(2) algebra

$$\Rightarrow \quad i_3 \in \{-i, -i+1, \ldots, i\}, \qquad 2i \in \mathbb{N}_0. \tag{4.10}$$

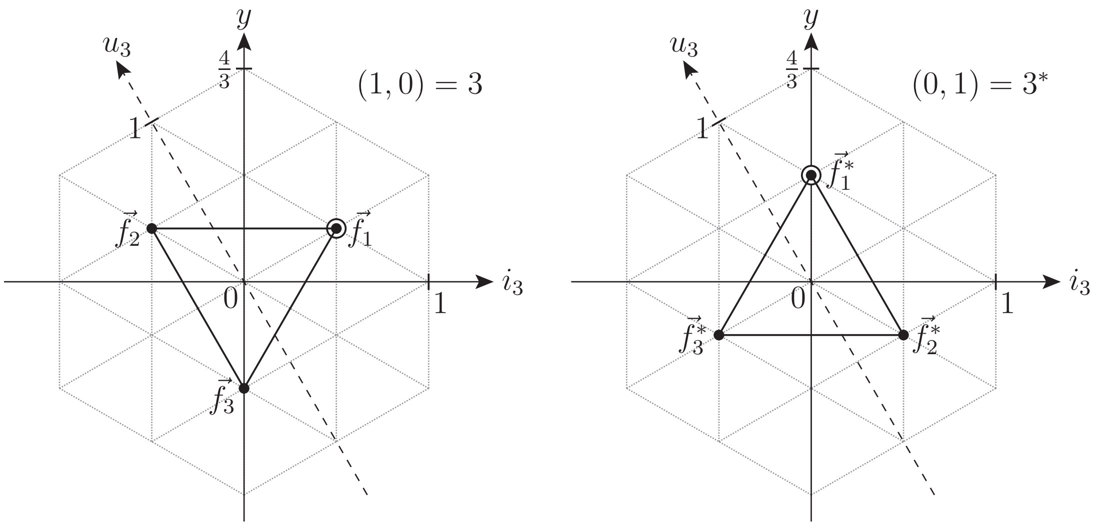
- U_1, U_2, $U_3 = \frac{1}{2}(I_3 + \frac{3}{2}Y)$ span an su(2) algebra

$$\begin{aligned} &\Rightarrow && u_3 = i_3 + \tfrac{3}{2}y \in \mathbb{Z} && && (4.11)\\ &\hookrightarrow && \tfrac{3}{2}y \in \mathbb{Z} \;\; (y = \ldots, -\tfrac{4}{3}, -\tfrac{2}{3}, 0, \tfrac{2}{3}, -\tfrac{4}{3}, \ldots) && \text{if } i_3 \text{ is integer,} \\ &\hookrightarrow && \tfrac{3}{2}(y + \tfrac{1}{3}) \in \mathbb{Z} \;\; (y = \ldots, -\tfrac{5}{3}, -1, -\tfrac{1}{3}, \tfrac{1}{3}, 1, \tfrac{5}{3}, \ldots) && \text{if } i_3 \text{ is half-integer.} \end{aligned}$$

  Choosing U_3 and $I_3 + \frac{1}{2}Y$ as diagonal basis elements instead shows that

$$u_3 \in \{-u, -u+1, \ldots, u\}, \qquad 2u \in \mathbb{N}_0. \tag{4.12}$$

SU(3) has two irreducible representations of dimension 3 corresponding to the rank-1 tensors with one upper index, T^j, or one lower index, T_j. The conditions on i_3 and u_3 fix the two possible sets of weight vectors that furnish the 3-dimensional representations:



- This is called a *weight diagram*.

- Denoting by (n, m) the upper rank n and lower rank m representations, $(1, 0)$ (left diagram) is called the *fundamental* representation and $(0, 1)$ (right diagram) the *anti-fundamental* representation.

- These are the lowest-dimensional non-trivial representations of SU(3).

- The encircled dot denotes the *highest weight* $\vec{\omega}_{\max} = \vec{f}_1^{(*)}$ of the representation, satisfying

$$I_+|\vec{\omega}_{\max}\rangle = U_+|\vec{\omega}_{\max}\rangle = V_+|\vec{\omega}_{\max}\rangle = 0, \tag{4.13}$$

  i.e. the root vectors corresponding to positive roots lead out of the representation's weight space.

- The weights can be constructed from the highest weight of the representation by applying I_- and U_- (in the corresponding representations). Fundamental representation:

$$\begin{aligned} &I_-|\vec{f}_1\rangle = |\vec{f}_2\rangle, && I_-|\vec{f}_2\rangle = 0, && I_-|\vec{f}_3\rangle = 0, \\ &U_-|\vec{f}_1\rangle = 0, && U_-|\vec{f}_2\rangle = |\vec{f}_3\rangle, && U_-|\vec{f}_3\rangle = 0. \end{aligned} \tag{4.14}$$

  Anti-fundamental representation:

$$\begin{aligned} &I_-|\vec{f}_1^*\rangle = 0, && I_-|\vec{f}_2^*\rangle = |\vec{f}_3^*\rangle, && I_-|\vec{f}_3^*\rangle = 0, \\ &U_-|\vec{f}_1^*\rangle = |\vec{f}_2^*\rangle, && U_-|\vec{f}_2^*\rangle = 0, && U_-|\vec{f}_3^*\rangle = 0. \end{aligned} \tag{4.15}$$

Constructing higher-dimensional irreducible representations

The irreducible representation (n, m) can be constructed from its highest weight vector

$$|\tfrac{n}{2}, \tfrac{1}{3}(n+2m)\rangle = \underbrace{|\vec{f}_1\rangle \otimes \cdots \otimes |\vec{f}_1\rangle}_{n \text{ times}} \otimes \underbrace{|\vec{f}_1^*\rangle \otimes \cdots \otimes |\vec{f}_1^*\rangle}_{m \text{ times}} \tag{4.16}$$

by applying the root vectors

$$\begin{aligned} I_-^{(n,m)} &= I_- \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \;+\; \mathbb{1} \otimes I_- \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \;+\; \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes I_-, \\ U_-^{(n,m)} &= U_- \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \;+\; \mathbb{1} \otimes U_- \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \;+\; \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes U_- \end{aligned} \tag{4.17}$$

in this representation.

Example: The irreducible representation (1,1)

Apply I_- ($\swarrow$) and U_- ($\searrow$) repeatedly to the highest weight $|\tfrac{1}{2}, 1\rangle = |\vec{f}_1\rangle \otimes |\vec{f}_1^*\rangle$ (omitting $\otimes$ in the following).

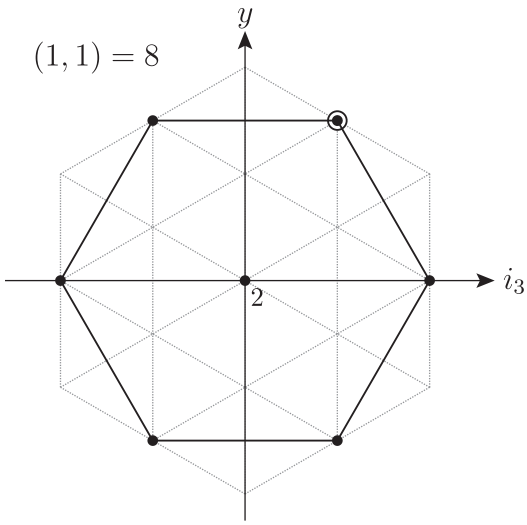
$$\begin{array}{ccccccccccccc}
 & & & & & & |\vec{f}_1\rangle|\vec{f}_1^*\rangle & & & & & & \\
 & & & & & \swarrow & & \searrow & & & & & \\
 & & & |\vec{f}_2\rangle|\vec{f}_1^*\rangle & & & & & & |\vec{f}_1\rangle|\vec{f}_2^*\rangle & & & \\
 & & \swarrow & & \searrow & & & & \swarrow & & \searrow & & \\
 & 0 & & |\vec{f}_3\rangle|\vec{f}_1^*\rangle + |\vec{f}_2\rangle|\vec{f}_2^*\rangle & & & & & |\vec{f}_2\rangle|\vec{f}_2^*\rangle + |\vec{f}_1\rangle|\vec{f}_3^*\rangle & & 0 & & \\
 & & \swarrow & & \searrow & & & & \swarrow & & \searrow & & \\
 & |\vec{f}_2\rangle|\vec{f}_3^*\rangle & & & 2|\vec{f}_3\rangle|\vec{f}_2^*\rangle & & & & 2|\vec{f}_2\rangle|\vec{f}_3^*\rangle & & |\vec{f}_3\rangle|\vec{f}_2^*\rangle & & \\
 \swarrow & & \searrow & \swarrow & & \searrow & & \swarrow & & \searrow & \swarrow & & \searrow \\
0 & & |\vec{f}_3\rangle|\vec{f}_3^*\rangle & 2|\vec{f}_3\rangle|\vec{f}_3^*\rangle & & & 0 & & & 2|\vec{f}_3\rangle|\vec{f}_3^*\rangle & |\vec{f}_3\rangle|\vec{f}_3^*\rangle & & 0
\end{array}$$

- There are 8 different states.
- The two states $|\vec{f}_3\rangle|\vec{f}_1^*\rangle + |\vec{f}_2\rangle|\vec{f}_2^*\rangle$ and $|\vec{f}_2\rangle|\vec{f}_2^*\rangle + |\vec{f}_1\rangle|\vec{f}_3^*\rangle$ have the same weight (indicated by the multiplicity 2 next to the weight), because

$$\vec{f}_1 + \vec{f}_3^* = \vec{f}_2 + \vec{f}_2^* = \vec{f}_3 + \vec{f}_1^* = (0,0).$$

- $\exists$ a 3rd linear combination $|\vec{f}_2\rangle|\vec{f}_2^*\rangle - |\vec{f}_1\rangle|\vec{f}_3^*\rangle - |\vec{f}_3\rangle|\vec{f}_1^*\rangle$ of weight $|0,0\rangle$ that does not belong to the representation $(1,1)$. This must be the representation $(0,0)$:

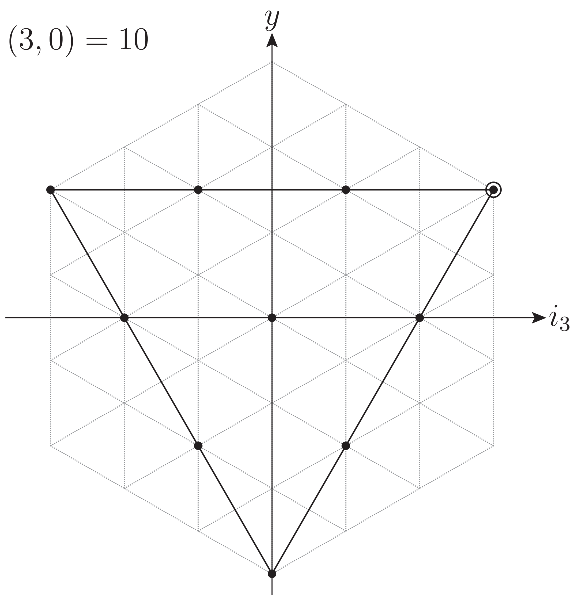
$$\begin{array}{ccccccc} (1,0) & \otimes & (0,1) & = & (1,1) & \oplus & (0,0), \\ 3 & \otimes & 3^* & = & 8 & \oplus & 1. \end{array}$$



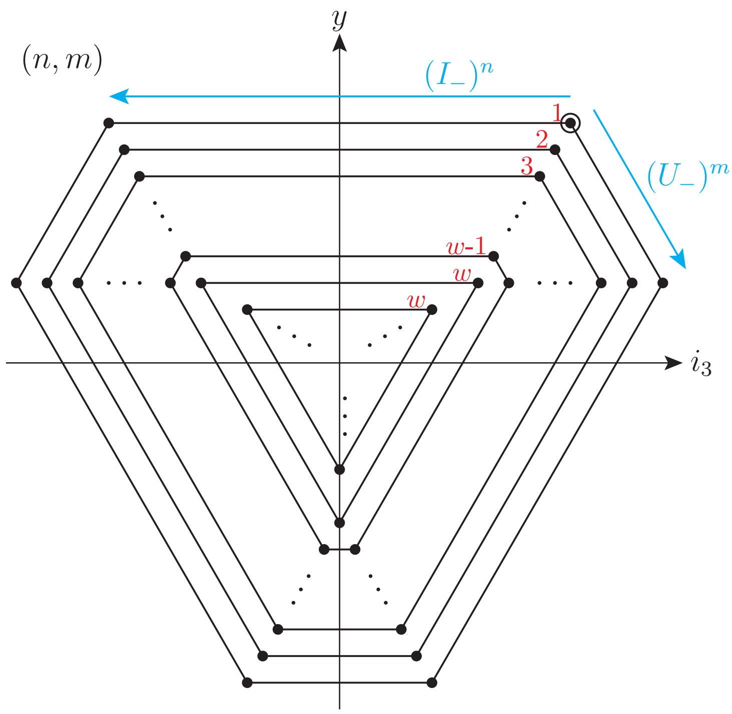
Example: The weight diagram of the representation (3,0)

- Start from highest weight $|\vec{f}_1\rangle|\vec{f}_1\rangle|\vec{f}_1\rangle = |\frac{3}{2}, 1\rangle$.
- 10 states, no multiple weights.
- Highest-dimensional representation in the Clebsch-Gordan series

$$\begin{array}{ccccccccccccc} (1,0) & \otimes & (1,0) & \otimes & (1,0) & = & (3,0) & \oplus & (1,1) & \oplus & (1,1) & \oplus & (0,0), \\ 3 & \otimes & 3 & \otimes & 3 & = & 10 & \oplus & 8 & \oplus & 8 & \oplus & 1. \end{array}$$



General case: The weight diagram of a representation (n,m)



Weight multiplicities:

- Red numbers in the diagram.
- Weights on the outermost hexagon have multiplicity 1. Multiplicity increases by 1 on each hexagon closer to the origin, but stays constant at maximal multiplicity $w = \min(n, m) + 1$ once the hexagon turns into a triangle.
- Multiplicities can, e.g., be calculated by Freudenthal's formula (see Section 6.5.3).
- Dimension of the representation:

$$\dim(n, m) = \frac{1}{2}(n+1)(m+1)(n+m+2)$$

as derived in Section 3.7.

4.3 Clebsch-Gordan decomposition

This section lists results and recipes. For more information see, e.g., *M. Grigorescu: SU(3) Clebsch-Gordan Coeffixients (arXiv:math-ph/0007033).*

Besides n, m, i_3, y, one more label is required to distinguish degenerate weights. This can be achieved by $\vec{I}^2 = I_1^2 + I_2^2 + I_3^3$ with eigenvalues $i(i+1)$:

$$\begin{aligned}\vec{I}^2|n,m,i,i_3,y\rangle &= i(i+1)|n,m,i,i_3,y\rangle,\\ I_3|n,m,i,i_3,y\rangle &= i_3|n,m,i,i_3,y\rangle,\\ Y|n,m,i,i_3,y\rangle &= y|n,m,i,i_3,y\rangle.\end{aligned} \tag{4.18}$$

i can take values $2i \in \mathbb{N}_0$ with

$$\left|\tfrac{1}{3}(m-n) - \tfrac{1}{2}y\right| \le i \le i_{\max}, \qquad i_{\max} = \begin{cases} \frac{1}{3}(2n+m) - \frac{1}{2}y & \text{if} \quad y \ge \frac{1}{3}(n-m), \\ \frac{1}{3}(n+2m) + \frac{1}{2}y & \text{if} \quad y \le \frac{1}{3}(n-m). \end{cases} \tag{4.19}$$

The operators corresponding to n and m are the two Casimir operators

$$C_1 = \sum_a T_a T_a, \qquad C_2 = \sum_{a,b,c} d_{abc} T_a T_b T_c \tag{4.20}$$

that have the form

$$\begin{aligned} C_1 &= \Big(\frac{1}{3}(n^2 + nm + m^2) + n + m\Big)\mathbb{1},\\ C_2 &= \frac{1}{18}(n-m)(n+2m+3)(m+2n+3)\,\mathbb{1} \end{aligned} \tag{4.21}$$

in the representation (n,m). C_1, C_2, $\vec{I}^2$, I_3, Y form a complete set of commuting operators. $I_\pm$, $U_\pm$, $V_\pm$ act as

$$I_\pm|n,m,i,i_3,y\rangle = \sqrt{i(i+1) - i_3(i_3 \pm 1)}|n,m,i,i_3 \pm 1,y\rangle, \tag{4.22}$$

$$\begin{aligned} U_+|n,m,i,i_3,y\rangle = &+ \gamma^+_{n,m,i,i_3,y}|n,m,i+\tfrac{1}{2},i_3-\tfrac{1}{2},y+1\rangle \\ &- \gamma^-_{n,m,i,i_3,y}|n,m,i-\tfrac{1}{2},i_3-\tfrac{1}{2},y+1\rangle, \end{aligned} \tag{4.23}$$

$$\begin{aligned} U_-|n,m,i,i_3,y\rangle = &- \gamma^-_{n,m,i+\frac{1}{2},i_3+\frac{1}{2},y-1}|n,m,i+\tfrac{1}{2},i_3+\tfrac{1}{2},y-1\rangle \\ &+ \gamma^+_{n,m,i-\frac{1}{2},i_3+\frac{1}{2},y-1}|n,m,i-\tfrac{1}{2},i_3+\tfrac{1}{2},y-1\rangle, \end{aligned} \tag{4.24}$$

$$\begin{aligned} V_+|n,m,i,i_3,y\rangle = &+ \gamma^+_{n,m,i,-i_3,y}|n,m,i+\tfrac{1}{2},i_3+\tfrac{1}{2},y+1\rangle \\ &+ \gamma^-_{n,m,i,-i_3,y}|n,m,i-\tfrac{1}{2},i_3+\tfrac{1}{2},y+1\rangle, \end{aligned} \tag{4.25}$$

$$\begin{aligned} V_-|n,m,i,i_3,y\rangle = &+ \gamma^-_{n,m,i+\frac{1}{2},-i_3+\frac{1}{2},y-1}|n,m,i+\tfrac{1}{2},i_3-\tfrac{1}{2},y-1\rangle \\ &+ \gamma^+_{n,m,i-\frac{1}{2},-i_3+\frac{1}{2},y-1}|n,m,i-\tfrac{1}{2},i_3-\tfrac{1}{2},y-1\rangle, \end{aligned} \tag{4.26}$$

with

$$\begin{aligned} \gamma^-_{n,m,i,i_3,y} =& \sqrt{\frac{i+i_3}{2i(2i+1)}} \\ &\times \sqrt{(\tfrac{1}{3}(2n+m) + i - \tfrac{1}{2}y + 1)(\tfrac{1}{3}(n+2m) - i + \tfrac{1}{2}y + 1)(\tfrac{1}{3}(m-n) + i - \tfrac{1}{2}y)}, \\ \gamma^+_{n,m,i,i_3,y} =& \sqrt{\frac{3+2i}{1+2i}}\,\gamma^-_{m,n,i+1,-i_3,-y}. \end{aligned} \tag{4.27}$$

Clebsch-Gordan coefficients

Tensor product of representations (n_1, m_1) and (n_2, m_2) (see Section 3.7):

$$(n_1, m_1) \otimes (n_2, m_2) = \bigoplus_k (n^k, m^k). \tag{4.28}$$

Express product states

$$|n_1, m_1, i_1, i_{1,3}, y_1; n_2, m_2, i_2, i_{2,3}, y_2\rangle \equiv |n_1, m_1, i_1, i_{1,3}, y_1\rangle |n_2, m_2, i_2, i_{2,3}, y_2\rangle, \tag{4.29}$$

which are eigenstates of

$$C_{1,1}, \quad C_{1,2}, \quad \vec{I}_1^{\,2}, \quad I_{1,3}, \quad Y_1, \quad C_{2,1}, \quad C_{2,2}, \quad \vec{I}_2^{\,2}, \quad I_{2,3}, \quad Y_2, \tag{4.30}$$

in terms of

$$|n^k, m^k, i^k, i_3^k, y^k\rangle_\gamma \tag{4.31}$$

which are eigenstates of

$$\begin{aligned} &C_1, \quad C_2, \quad C_{1,1}, \quad C_{1,2}, \quad C_{2,1}, \quad C_{2,2}, \\ &\vec{I}^2 = \left(\vec{I}_1 + \vec{I}_2\right)^2, \quad I_3 = I_{1,3} + I_{2,3}, \quad Y = Y_1 + Y_2. \end{aligned} \tag{4.32}$$

There are 10 operators in (4.30), but only 9 in (4.32). This reflects the fact that the same representation may appear multiply on the right-hand side of (4.28) and is taken into account by the index γ in (4.31). It is possible to find an operator to complete the set (4.32), but it is more convenient to use an orthogonalisation procedure instead.

1. Start with the subspace of highest weight in (4.28) and apply I_- and U_- to calculate all states in this space.

2. Proceed to the subspaces with the next-to-highest weight, which have all the same highest weight. If there is more than one subspace with this highest weight, choose states so that

$$_\gamma\langle n^k, m^k, i^k, i_3^k, y^k | n^k, m^k, i^k, i_3^k, y^k\rangle_{\gamma'} = \delta_{\gamma\gamma'}. \tag{4.33}$$

3. Apply I_- and U_- to calculate all states in these spaces.

4. If there are any (combinations of) product states left, proceed with 2 for the next-to-next-to-highest weight, etc..

The Clebsch-Gordan coefficients then follow from

$$\begin{aligned} |n^k, m^k, i^k, i_3^k, y^k\rangle_\gamma = \sum_{i_1, i_2} \sum_{i_{1,3}, i_{2,3}} \sum_{y_1, y_2} &\langle n_1, m_1, i_1, i_{1,3}, y_1; n_2, m_2, i_2, i_{2,3}, y_2 | n^k, m^k, i^k, i_3^k, y^k\rangle_\gamma \\ &\times |n_1, m_1, i_1, i_{1,3}, y_1; n_2, m_2, i_2, i_{2,3}, y_2\rangle. \end{aligned} \tag{4.34}$$

4.4 Isospin and hypercharge

4.4.1 SU(2) isospin

Hadrons (= strongly interacting particles) occur in sets of similar mass of $\mathcal{O}(1\%)$ differences.

Nucleons: $m_p = 938.3\,\mathrm{MeV}/c^2$, $m_n = 939.6\,\mathrm{MeV}/c^2$ $\Rightarrow$ $\frac{m_n - m_p}{m_n + m_p} \approx 0.069\,\%$.

Pions: $m_{\pi^\pm} = 139.6\,\mathrm{MeV}/c^2$, $m_{\pi^0} = 135.0\,\mathrm{MeV}/c^2$ $\Rightarrow$ $\frac{m_{\pi^\pm} - m_{\pi^0}}{m_{\pi^\pm} + m_{\pi^0}} \approx 1.7\,\%$.

The strong interaction seems not to distinguish between particles in such a set.
↪ Hypothesis: Strong interaction is (approximately) invariant under an SU(2) "isospin" symmetry that transforms hadrons into each other.

- Nucleons form an isospin $I = \frac{1}{2}$ doublet (p, n).
- Pions form an isospin $I = 1$ triplet (π^+, π^0, π^-).
- Masses are not equal.
  ↪ Symmetry is broken, e.g. by (but not only by) electromagnetic interaction, because the particles have different electric charges.
- Symmetry constrains strong interaction between particles.
  ↪ Clebsch-Gordan coefficients & Wigner-Eckart theorem.

Example: Ratio of deuteron production cross sections

The deuteron d (heavy hydrogen nucleus) is a bound state of a proton and a neutron.

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0 \quad \Rightarrow \quad d \text{ has either } I = 0 \text{ or } I = 1. \tag{4.35}$$

pp and nn bound states have not been observed. $\Rightarrow$ d must form an $I = 0$ singlet.

An example:

$$\frac{\sigma(p + p \to d + \pi^+)}{\sigma(p + n \to d + \pi^0)} = \frac{|\langle d, \pi^+ | \mathcal{T} | p, p \rangle|^2}{|\langle d, \pi^0 | \mathcal{T} | p, n \rangle|^2} \tag{4.36}$$

with a transition operator $\mathcal{T}$ of definite SU(2) transformation property.

Well-motivated assumption: $\mathcal{T}$ = scalar (otherwise no isospin conservation in reaction, i.e. more particles should appear).

↪ Clebsch-Gordan decomposition:

$$\begin{aligned}
|p, p\rangle &\equiv |\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle = |\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle = |1, 1\rangle \\
|p, n\rangle &\equiv |\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = |\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\big(|1, 0\rangle - |0, 0\rangle\big), \\
|d, \pi^+\rangle &\equiv |0, 0\rangle \otimes |1, 1\rangle = |1, 1\rangle, \\
|d, \pi^0\rangle &\equiv |0, 0\rangle \otimes |1, 0\rangle = |1, 0\rangle.
\end{aligned} \tag{4.37}$$

$$\Rightarrow \quad \frac{\sigma(p + p \to d + \pi^+)}{\sigma(p + n \to d + \pi^-)} = \frac{|\langle 1, 1 | \mathcal{T} | 1, 0 \rangle|^2}{|\langle 1, 0 | \mathcal{T} \frac{1}{\sqrt{2}}\big(|1, 0\rangle - |0, 0\rangle\big)|^2} = 2. \tag{4.38}$$

Tensor method and effective field theory

Write nucleons as a vector N^i and pions as a rank-(1,1) tensor Φ^i_j,

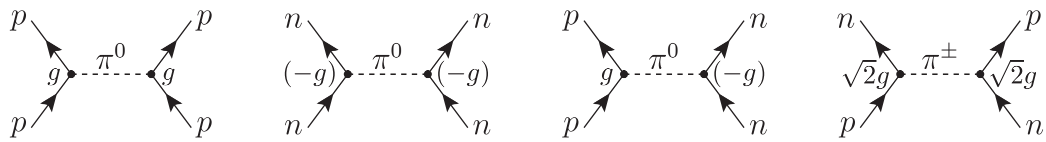
$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad \Phi = \vec{\pi} \cdot \frac{\vec{\sigma}}{2} = \begin{pmatrix} \pi_3 & \pi_1 - \mathrm{i}\pi_2 \\ \pi_1 + \mathrm{i}\pi_2 & -\pi_3 \end{pmatrix} \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi_0 \end{pmatrix}, \tag{4.39}$$

where $\vec{\pi} = (\pi_1, \pi_2, \pi_3)^{\mathrm{T}}$ is in the cartesian vector representation and (π^+, π^0, π^-) in the spherical basis.

$\hookrightarrow$ Build an SU(2)-invariant interaction Lagrangian of an effective theory of nucleons and pions by combining N and Φ to singlets (trivial representation):

$$\mathcal{L}_{\mathrm{int}} = g N_j \Phi^j_i N^i = g \bar{N} \Phi N = g\, \bar{p}\pi^0 p - g\, \bar{n}\pi^0 n + \sqrt{2}g\, \bar{p}\pi^+ n + \sqrt{2}g\, \bar{n}\pi^- p \tag{4.40}$$

with some coupling constant g. Feynman diagrams of nucleon scattering:



$\Rightarrow$ Relations between different (pp, np, $p\pi^0$, etc.) scattering cross sections can be derived.

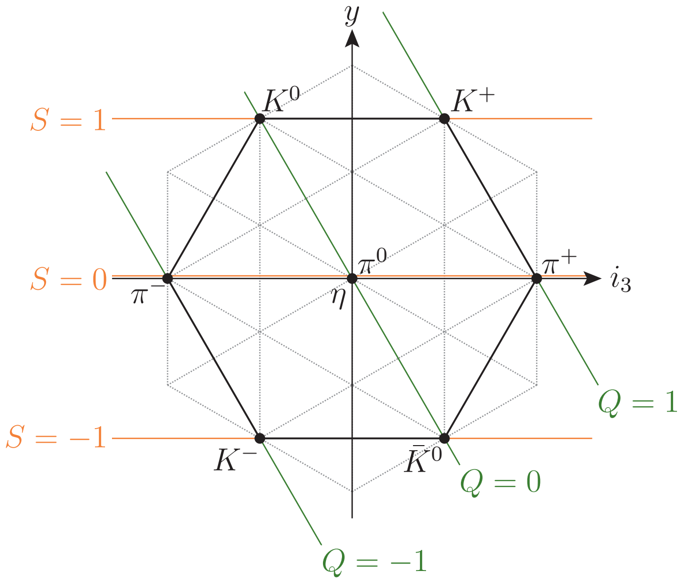
4.4.2 SU(3) flavour symmetry

Further experimental observations:

- Different SU(2) multiplets of hadrons of the same spin show typical mass differences by $\mathcal{O}(10\%)$ (for baryons) or more (for mesons).
- Some hadrons have longer lifetimes than expected from the strong interaction. $\hookrightarrow$ Explanation by the quantum number *"strangeness"* S that is conserved by the strong interaction. Those hadrons decay via the weak interaction.

$\Rightarrow$ SU(2) multiplets of hadrons of the same spin can be arranged into representations of the SU(3) flavour symmetry.

Spin-0 mesons:



- The octet consists of the pion triplet, the two kaon doublets (K^0, K^+) and (K^-, $\bar{K}^0$), and the isospin singlet η.
- This scheme of organising hadrons is called *"The Eightfold Way"*.
- Together with the η' in the $(0,0)$ representation, the spin-0 mesons form the $(1,0)\otimes(0,1)$ nonet.
- Electric charge: $Q = I_3 + \frac{1}{2}Y$ (Gell-Mann–Nishijima formula).
- Strangeness: $S = Y - B$ with the *"baryon number"* $B = 0$ for mesons.

Quarks and anti-quarks

This structure is explained by regarding hadrons as composite particles that consist of more fundamental particles called *quarks* and their anti-particles, *anti-quarks*, which furnish the fundamental rsp. anti-fundamental representations of SU(3).

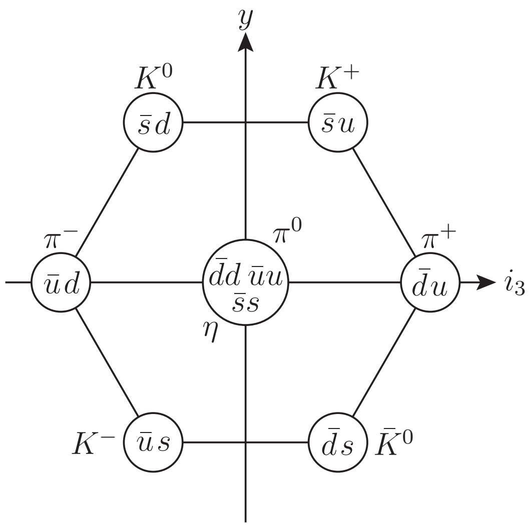
Quantum numbers of the u (*"up"*), d (*"down"*), and s (*"strange"*) quarks:

	Q	I	I_3	Y	S	B
u	$2/3$	$1/2$	$1/2$	$1/3$	0	$1/3$
d	$-1/3$	$1/2$	$-1/2$	$1/3$	0	$1/3$
s	$-1/3$	0	0	$-2/3$	-1	$1/3$

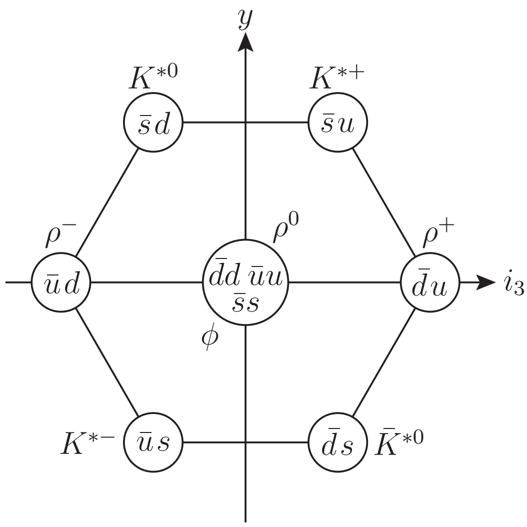
Differences in the quark masses are another source for breaking the flavour symmetry.

There are 3 more quarks (c=*"charm"*, b=*"bottom"*, t=*"top"*), but their masses are so large that the approximate flavour symmetries SU(4) and SU(5) are crudely broken. The top-quark does not even form bound states.

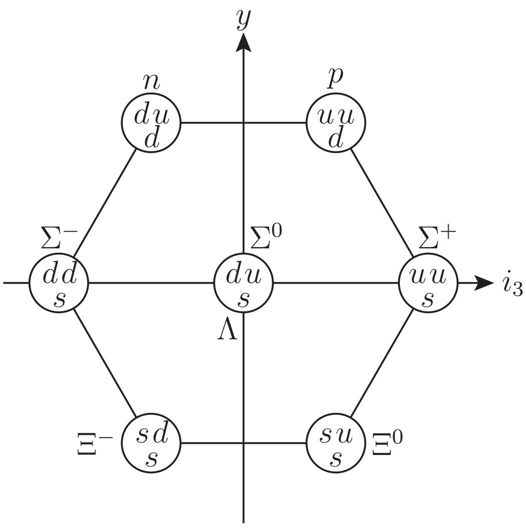
Baryon multiplets and triality



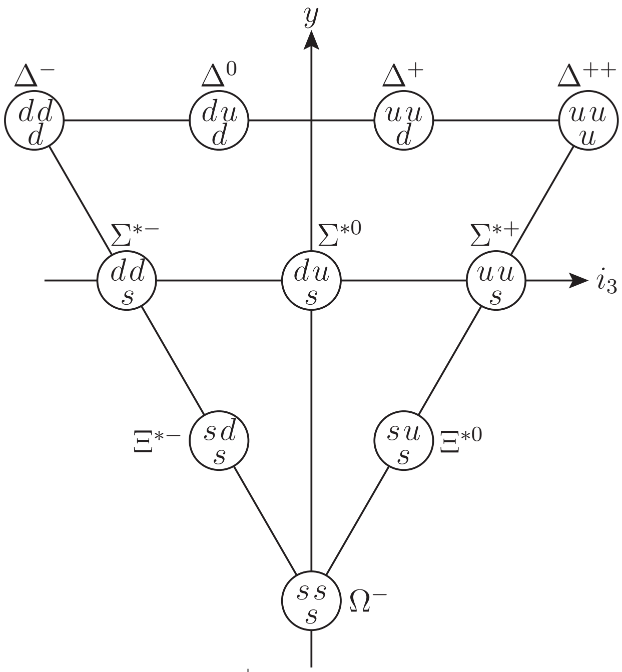
$J^P = 0^-$ pseudoscalar meson octet;
(0,0) representation: η'



$J^P = 1^-$ vector meson octet;
(0,0) representation: ω



$J^P = \frac{1}{2}^+$ baryon octet



$J^P = \frac{3}{2}^+$ baryon decuplet

Since quarks are fermions, the wave functions of hadrons must be totally antisymmetric under exchange of two quarks.
$\hookrightarrow$ How is this possible e.g. in the case of the spin-$\frac{3}{2}$ baryon Δ^{++} of 3 up quarks?

$$|\Delta^{++}\rangle = |u\uparrow\rangle|u\uparrow\rangle|u\uparrow\rangle \tag{4.41}$$

is totally symmetric.
$\Rightarrow$ There must exist another quantum number. This is the *"colour charge"*:

- 3 charges that transform under an SU(3) symmetry.
- This is the symmetry of quantum chromodynamics.
- Unlike flavour-SU(3), colour symmetry is exact.

Observable states must be colour singlets (*"colour confinement"*). This is the reason why only representations (n, m) with $n - m = 0 \pmod 3$ are populated with hadrons. This fact is called *triality*.

4.4.3 Gell-Mann–Okubo mass formula

The hadron octets can be arranged into the components of a tensor Φ^i_j. Spin-0 mesons:

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}. \tag{4.42}$$

Assuming exact flavour symmetry, the mass term in the Lagrangian would be

$$\mathcal{L}^{(0)}_{\text{mass}} = \frac{1}{2}\, m_1^2 \operatorname{Tr}\Phi^2 = \frac{1}{2}\, m_1^2 \Phi^i_j \Phi^j_i. \tag{4.43}$$

This would imply that all masses are equal. The symmetry can be broken by introducing mass terms that transform like the $(1,1)$ and the $(2,2)$ representations:

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}\, m_1^2 \Phi^i_j \Phi^j_i + \frac{1}{2}\, \Phi^i_j M^j_k \Phi^k_i + \frac{1}{2}\, \Phi^i_j \tilde{M}^{jl}_{ik} \Phi^k_l. \tag{4.44}$$

- *Assumption:* The SU(3) symmetry is only broken by the octet M^j_k, i.e. $\tilde{M}^{jl}_{ik} = 0$.
- The mass term must conserve i_3 and y.
  $\Rightarrow$ M^j_k transforms like the η meson
  $\Rightarrow$ $M = 3m_2^2 Y$ (factor 3 is convention).

The mass term is thus (note that $\bar{K}^0$ is the antiparticle of K^0 and K^- that of K^+)

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= \frac{1}{2}\, m_1^2 \operatorname{Tr}\Phi^2 + \frac{1}{2} \operatorname{Tr}\Phi M \Phi = \frac{1}{2}\, m_\eta^2 \eta^2 + \frac{1}{2}\, m_\pi^2 \operatorname{Tr}\bar{\pi}\pi + \frac{1}{2}\, m_K^2 \bar{K} K \\ \text{with} \quad & m_\eta^2 = m_1^2 - m_2^2, \qquad m_\pi^2 = m_1^2 + m_2^2, \qquad m_K^2 = m_1^2 - \frac{1}{2} m_2^2. \end{aligned} \tag{4.45}$$

Eliminating m_1 and m_2 in (4.45) shows that

$$4m_K^2 = 3m_\eta^2 + m_\pi^2, \tag{4.46}$$

which is fulfilled to better than 4 %.

With the same method, a mass formula for the baryon octet can be derived, where two symmetry breaking terms transforming under (1,1) can appear: $m_2 \operatorname{Tr}\bar{B}YB$ and $m_3 \operatorname{Tr}\bar{B}BY$. Instead of working this out, we derive a formula for the case of a hadron multiplet of an arbitrary representation of SU(3).

There can be at most two symmetry breaking mass terms that transform under the (1,1) representation. For a baryon multiplet $B^{i_1 \ldots i_n}_{j_1 \ldots j_m}$ of (m,n):

$$\bar{B}^{j_1 \ldots j_m}_{i_1 \ldots i_n} \phi^{i_1}_k B^{k i_2 \ldots i_n}_{j_1 \ldots j_m}, \qquad \bar{B}^{k j_2 \ldots j_m}_{i_1 \ldots i_n} \phi^{j_1}_k B^{i_1 \ldots i_n}_{j_1 \ldots j_m}, \tag{4.47}$$

with some tensor ϕ^i_j.
$\Rightarrow$ Expressing the mass terms in terms of operators acting on the hadron multiplets, there

can be at most two such operators.

The generators of a Lie algebra transform under the adjoint representation.
↪ Arrange the generators of SU(3) in a traceless 3×3 matrix G:

$$G = \begin{pmatrix} I_3 + \frac{1}{2}Y & I_- & V_- \\ I_+ & -I_3 + \frac{1}{2}Y & U_- \\ V_+ & U_+ & -Y \end{pmatrix} . \tag{4.48}$$

From the same arguments as in the case of the meson octet, one of the possible operators is Y, i.e. the component G^3_3. The second operator can be constructed by projecting out an octet in the Clebsch-Gordan decomposition of the tensor product $G^i_j G^l_m$,

$$\begin{aligned} \tilde{G}^b_a &= \frac{1}{2}\epsilon_{ajl}\epsilon^{bkm} G^j_k G^l_m \\ \Rightarrow \quad \tilde{G}^3_3 &= \frac{1}{2}\big(G^1_1 G^2_2 + G^2_2 G^1_1 - G^1_2 G^2_1 - G^2_1 G^1_2\big) \\ &= \frac{1}{4}Y^2 - I_3^2 - \frac{1}{2}(I_+ I_- + I_- I_+) = \frac{1}{4}Y^2 - \vec{I}^2 . \end{aligned} \tag{4.49}$$

Note that $\tilde{G}^b_a$ is not yet traceless, but this does not affect the mass formula. The masses of the particles in a SU(2) multiplet of isospin i and hypercharge y are thus

$$M_{i,y} = m_1 + m_2 y + m_3\big(\tfrac{1}{4}y^2 - i(i+1)\big) \tag{4.50}$$

with parameters m_1, m_2, m_3. This is the Gell-Mann–Okubo mass formula.

In case of the baryon octet we obtain

$$\begin{aligned} m_N &\equiv M_{\frac{1}{2},1} = m_1 + m_2 - \frac{1}{2}m_3, & m_\Lambda &\equiv M_{0,0} = m_1, \\ m_\Xi &\equiv M_{\frac{1}{2},-1} = m_1 - m_2 - \frac{1}{2}m_3, & m_\Sigma &\equiv M_{1,0} = m_1 - 2m_3 \\ \Rightarrow \quad & m_\Sigma + 3m_\Lambda = 2m_N + 2m_\Xi . \end{aligned} \tag{4.51}$$

This relation is fulfilled to better than 3 ‰.

Comment: In a similar way it is possible to derive relations between magnetic moments of hadrons (though not as a generic formula for arbitrary representations).

Chapter 5

Lie groups and Lie algebras

5.1 Lie groups

Definitions:

- "Lie group" $\equiv$ a smooth manifold G that is also a group with the property that the group product $G \times G \to G$ and the inverse map $G \to G : g \mapsto g^{-1}$ are smooth.

  Loosely speaking, a "smooth manifold" is a set of points that looks locally like a neighbourhood of some point of $\mathbb{R}^n$, and "smooth" mappings are meant to be infinitely many times differentiable (for precise definitions, see, e.g., Ref. [2]).

- "Matrix Lie group" $\equiv$ closed subgroup of $\mathrm{GL}(\mathbb{C}^n)$.

  "Closed" means here: If $\{A_m\}$ is some sequence of matrices in G converging to some matrix A, then either $A \in G$ or A is not invertible.

This lecture focuses on matrix Lie groups:

- do not exhaust all Lie groups, but by far the most important in physics;
- are easier to handle (manipulations made very explicit).

Examples for groups that are not Lie groups:

- $\mathrm{GL}(\mathbb{Q}^n)$ = invertible $n \times n$ matrices with coefficients $\in \mathbb{Q}$.
- $G = \{ \mathrm{diag}(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}at}) \,|\, t \in \mathbb{R} \}$, with fixed $a \in \mathbb{R}$, but $a \notin \mathbb{Q}$.

For an example of a Lie group that is not a matrix Lie group and has no faithful finite-dimensional representations, see chap. 4.8 in [6].

Characterization of a Lie group G

- Group multiplication encoded in analytical mappings $f_A(\vec{\theta}\,', \vec{\theta}\,)$ of group parameters $\vec{\theta}\,', \vec{\theta}$:

$$\begin{aligned} g'' &= g'g, \quad g(\vec{\theta}\,'') = g(\vec{\theta}\,')\, g(\vec{\theta}\,), \quad g, g', g'' \in G, \\ \theta''_A &= f_A(\vec{\theta}\,', \vec{\theta}\,), \quad A = 1, \dots, n = \dim G, \\ \theta_A &= f_A(0, \vec{\theta}\,) = f_A(\vec{\theta}\,, 0), \quad \text{since } g(\vec{0}\,) = e. \end{aligned} \tag{5.1}$$

  The existence of $g(\vec{\theta}\,)^{-1}$, in particular, implies the local invertibility of f_A:

$$\begin{aligned} \Theta^B{}_A(\vec{\theta}\,) &\equiv \frac{\partial f_A(\vec{\theta}\,', \vec{\theta}\,)}{\partial \theta'_B}\Big|_{\vec{\theta}\,'=\vec{0}} = \text{non-singular}, \quad \Theta(\vec{0}) = \mathbb{1}, \\ \Psi(\vec{\theta}\,) &\equiv \Theta(\vec{\theta}\,)^{-1}. \end{aligned} \tag{5.2}$$

- *Locally* a Lie group is fully determined by its “Lie algebra” (Lie's theorems).
  $\hookrightarrow$ General Lie groups treated below!

  Special case of matrix Lie groups (previous chapters):
  Lie algebra spanned by the generators T^A for infinitesimal group elements

$$U(\delta\vec{\theta}\,) = \mathbb{1} - \mathrm{i}\delta\theta_A T^A + \mathcal{O}(\delta\theta_A^2), \tag{5.3}$$

  with the commutators $[T^A, T^B] = T^A T^B - T^B T^A$ as product of generators.
  Note: In general Lie algebras there is no matrix multiplication to define $T^A T^B$.

- *Global* properties of the group parameter space are necessary to define a Lie group uniquely.

Important global properties:

- "Compactness": Group parameter space is compact in the topological sense.

  Compact groups have similar properties as finite groups, in particular wrt. representation theory (finite-dim. representation can be chosen unitary).

  $\hookrightarrow$ Finite representations can directly represent qm. states.

  Examples:

  - Compact: O(N), SO(N), U(N), SU(N).
  - Non-compact: translational group, Euclidean groups, Lorentz group.

- "Connectedness": Each element is connected to the identity element by a continuous path in G.

  $\hookrightarrow$ Group parameter space decomposes into disjoint, isomorphic sets G_j, $G = \cup_j G_j$, but only one component (the "identity component" G_0) contains the unit element.

  Some properties:

  - The components $G_{j\neq 0}$ are no groups ($e \notin G_{j\neq 0}$).
  - G_0 is an invariant subgroup of G.
  - The factor group $D_G = G/G_0$ is a (finite or infinite) discrete group.

  Examples:

  - Connected: SO(N), U(N), SU(N).
  - Not connected: O(N), Lorentz group.

- "Simple connectedness": Each closed path in G can be continuously contracted to a point.

  Each connected Lie group G has a "universal covering group" which is locally isomorphic (isomorphic Lie algebras) and simply connected.

  (Subtlely: The universal covering group of a mtrix Lie group might not be a matrix Lie group.)

  If a Lie group has m independent non-equivalent closed curves ("m-connected group"), m-valued representations are possible.

  $\hookrightarrow$ Universal covering groups only have single-valued representations.

  Examples:

  - Simply connected: SU(N).
  - Not simply connected: SO(N).
  - Recall: SU(2) is universal covering group of SO(3).

Local properties (Lie's theorems and their converses)

In addition to the Lie group G itself, consider its realization as transformations on some vector $\vec{x} \in \mathbb{R}^N$:

$$x'_a = F_a(\vec{\theta}, \vec{x}), \quad x_a = F_a(\vec{0}, \vec{x}), \qquad a = 1, \ldots, N. \tag{5.4}$$

Infinitesimal trafo $\delta\vec{\theta}$ near identity ($\vec{\theta} = \vec{0}$):

$$x_a + \mathrm{d}x_a = F_a(\delta\vec{\theta}, \vec{x}), \quad \mathrm{d}x_a = \delta\theta_A \, u_a^A(\vec{x}), \quad u_a^A(\vec{x}) \equiv \frac{\partial F_a(\vec{\theta}, \vec{x})}{\partial \theta_A}\Big|_{\vec{\theta}=\vec{0}}. \tag{5.5}$$

Infinitesimal trafo $\mathrm{d}\vec{\theta}$ near finite $\vec{\theta}$: $\mathrm{d}\vec{\theta}$ and $\delta\vec{\theta}$ are related by $\theta_A + \mathrm{d}\theta_A = f_A(\delta\vec{\theta}, \vec{\theta})$.

$$\Rightarrow \; \mathrm{d}\theta_A = \delta\theta_B \, \Theta^B{}_A(\vec{\theta}), \qquad \delta\theta_B = \mathrm{d}\theta_A \, \Psi^A{}_B(\vec{\theta}) \tag{5.6}$$

according to (5.2).

$$\begin{aligned} \Rightarrow \; x'_a + \mathrm{d}x'_a &= F_a(\vec{\theta} + \mathrm{d}\vec{\theta}, \vec{x}) = F_a(\delta\vec{\theta}, \vec{x}\,'), \\ \mathrm{d}x'_a &= u_a^B(\vec{x}\,') \, \delta\theta_B = \mathrm{d}\theta_A \, \Psi^A{}_B(\vec{\theta}) \, u_a^B(\vec{x}\,'). \end{aligned} \tag{5.7}$$

Lie's theorems:

- Lie's 1st theorem:

$$\frac{\partial x'_a}{\partial \theta_A} = \Psi^A{}_B(\vec{\theta}) \, u_a^B(\vec{x}\,') \tag{5.8}$$

  with analytical functions $\Psi^A{}_B(\vec{\theta})$ and $u_a^B(\vec{x}\,')$.

  Note: decoupling of $\vec{\theta}$ and $\vec{x}\,'$ dependences in evolution in θ_A!

- Lie's 2nd theorem:

  The generators

$$\mathcal{X}^A(\vec{\theta}) \equiv -\mathrm{i}\Theta^A{}_B(\vec{\theta}) \, \frac{\partial}{\partial \theta_B}, \qquad X^A(\vec{x}) \equiv -\mathrm{i}u_a^A(\vec{x}) \, \frac{\partial}{\partial x_a} \tag{5.9}$$

  obey the commutation relations:

$$[\mathcal{X}^A(\vec{\theta}), \mathcal{X}^B(\vec{\theta})] = \mathrm{i}f^{AB}{}_C \, \mathcal{X}^C(\vec{\theta}), \qquad [X^A(\vec{x}), X^B(\vec{x})] = \mathrm{i}f^{AB}{}_C \, X^C(\vec{x}) \tag{5.10}$$

  with the "structure constants", which neither depend on $\vec{\theta}$ nor on $\vec{x}$.

- Lie's 3rd theorem:

  The structure constants obey

$$f^{AB}{}_C = -f^{BA}{}_C. \qquad \text{(antisymmetry)} \tag{5.11}$$

$$0 = f^{AB}{}_C f^{DC}{}_E + f^{DA}{}_C f^{BC}{}_E + f^{BD}{}_C f^{AC}{}_E. \qquad \text{(Jacobi identity)} \tag{5.12}$$

  Both equations immediately follow from the definitions of the generators, in particular the second is due to $[[\mathcal{X}^A(\vec{\theta}), \mathcal{X}^B(\vec{\theta})], \mathcal{X}^C(\vec{\theta})] + \text{cyclic} = 0$.

Proof of Lie's 2nd theorem:

Take derivative of (5.8) wrt. θ_C:

$$\begin{aligned}\frac{\partial^2 x'_a}{\partial\theta_A \partial\theta_C} &= \frac{\partial}{\partial\theta_C}\left[\Psi^A{}_B(\vec\theta)\, u^B_a\big(\vec x\,'(\vec\theta)\big)\right] \\ &= \frac{\partial\Psi^A{}_B(\vec\theta)}{\partial\theta_C}\, u^B_a\big(\vec x\,'(\vec\theta)\big) + \Psi^A{}_B(\vec\theta)\,\frac{\partial u^B_a}{\partial x'_b}\,\frac{\partial x'_b}{\partial\theta_C} \\ &= \frac{\partial\Psi^A{}_B(\vec\theta)}{\partial\theta_C}\, u^B_a\big(\vec x\,'(\vec\theta)\big) + \Psi^A{}_B(\vec\theta)\,\frac{\partial u^B_a}{\partial x'_b}\,\Psi^C{}_D(\vec\theta)\, u^D_b(\vec x\,'). \end{aligned} \tag{5.13}$$

Using $\frac{\partial^2 x'_a}{\partial\theta_A\partial\theta_C} = \frac{\partial^2 x'_a}{\partial\theta_C\partial\theta_A}$ and renaming indices, we get

$$\left(\frac{\partial\Psi^A{}_B(\vec\theta)}{\partial\theta_C} - \frac{\partial\Psi^C{}_B(\vec\theta)}{\partial\theta_A}\right) u^B_a(\vec x\,') = \Psi^A{}_B(\vec\theta)\,\Psi^C{}_D(\vec\theta)\left[\frac{\partial u^D_a}{\partial x'_b}\, u^B_b(\vec x\,') - \frac{\partial u^B_a}{\partial x'_b}\, u^D_b(\vec x\,')\right]. \tag{5.14}$$

Aim: separation of variables $\vec\theta$ and $\vec x\,'$, but problem with $u^B_a(\vec x\,')$ term on l.h.s., which is not necessarily invertible.

$\hookrightarrow$ Take special case for $x'_a = F_a(\vec\theta, \vec x)$ interpreting $\vec x\,'$ as $\vec\theta\,'$:

$$\vec x\,' \to \vec\theta\,', \qquad u^A_b(\vec x\,') \to \Theta^A{}_B(\vec\theta\,').$$

$$\begin{aligned}\Rightarrow\quad &\left(\frac{\partial\Psi^A{}_B(\vec\theta)}{\partial\theta_C} - \frac{\partial\Psi^C{}_B(\vec\theta)}{\partial\theta_A}\right)\Theta^B{}_E(\vec\theta\,') \\ &\quad = \Psi^A{}_B(\vec\theta)\,\Psi^C{}_D(\vec\theta)\left[\frac{\partial\Theta^D{}_E}{\partial\theta'_F}\,\Theta^B{}_F(\vec\theta\,') - \frac{\partial\Theta^B{}_E}{\partial\theta'_F}\,\Theta^D{}_F(\vec\theta\,')\right].\end{aligned}$$

$$\begin{aligned}\Leftrightarrow\quad &\underbrace{\Theta^H{}_A(\vec\theta)\,\Theta^I{}_C(\vec\theta)\left(\frac{\partial\Psi^A{}_G(\vec\theta)}{\partial\theta_C} - \frac{\partial\Psi^C{}_G(\vec\theta)}{\partial\theta_A}\right)}_{\text{function of } \vec\theta} \\ &\quad = \underbrace{\left[\frac{\partial\Theta^I{}_E}{\partial\theta'_F}\,\Theta^H{}_F(\vec\theta\,') - \frac{\partial\Theta^H{}_E}{\partial\theta'_F}\,\Theta^I{}_F(\vec\theta\,')\right]\Psi^E{}_G(\vec\theta\,')}_{\text{function of } \vec\theta\,'} \overset{!}{=} \text{const.} \equiv -f^{HI}{}_G.\end{aligned} \tag{5.15}$$

The remaining steps are fully straightforward:

- Calculate commutators of $\mathcal{X}^A(\vec\theta)$:

$$\begin{aligned}\left[\mathcal{X}^A(\vec\theta), \mathcal{X}^B(\vec\theta)\right] &= \left[-\mathrm{i}\Theta^A{}_C(\vec\theta)\,\frac{\partial}{\partial\theta_C}, -\mathrm{i}\Theta^B{}_D(\vec\theta)\,\frac{\partial}{\partial\theta_D}\right] \\ &= \underbrace{\left(-\Theta^A{}_C(\vec\theta)\,\frac{\partial\Theta^B{}_E(\vec\theta)}{\partial\theta_C} + \Theta^B{}_D(\vec\theta)\,\frac{\partial\Theta^A{}_E(\vec\theta)}{\partial\theta_D}\right)}_{= f^{AB}{}_F\,\Theta^F{}_E(\vec\theta)\ \text{accorcing to (5.15)}}\frac{\partial}{\partial\theta_E} = \mathrm{i} f^{AB}{}_F\,\mathcal{X}^F(\vec\theta).\end{aligned}$$

- Calculate commutators of $X^A(\vec{x})$:

$$\begin{aligned}\left[X^A(\vec{x}), X^A(\vec{x})\right] &= \left[-\mathrm{i}u_a^A(\vec{x})\,\frac{\partial}{\partial x_a}, -\mathrm{i}u_b^B(\vec{x})\,\frac{\partial}{\partial x_b}\right] \\ &= \left(-u_a^A(\vec{x})\,\frac{\partial u_c^B(\vec{x})}{\partial x_a} + u_b^B(\vec{x})\,\frac{\partial u_c^A(\vec{x})}{\partial x_b}\right)\frac{\partial}{\partial x_c} \\ &\underset{(5.14)}{=} \underbrace{\left(-\frac{\partial \Psi^C{}_E(\vec{\theta})}{\partial \theta_D} + \frac{\partial \Psi^D{}_E(\vec{\theta})}{\partial \theta_C}\right)\Theta^A{}_C(\vec{\theta})\,\Theta^B{}_D(\vec{\theta})}_{=f^{AB}{}_E \text{ accorcing to (5.15)}}\,u_c^E(\vec{x})\,\frac{\partial}{\partial x_c} \\ &= \mathrm{i}f^{AB}{}_E\,X^E(\vec{x}).\end{aligned}$$

#

Converse statements of Lie's theorems:

- Converse of the 1st theorem:

  If functions $f_A(\vec{\theta}\,', \vec{\theta})$ and $F_a(\vec{\theta}, \vec{x})$ that are analytic around $\vec{\theta} = \vec{\theta}\,' = \vec{0}$ and $\vec{x} = \vec{0}$ exist, then there is a corresponding "local Lie group" and "local Lie transformations" (i.e. in the vicinities of the group identity and of points $\vec{x} = \vec{0}$) with the generators $\mathcal{X}^A(\vec{\theta})$ and $X^A(\vec{x})$.

- Converse of the 2nd theorem:

  The Lie algebra of the generators $\mathcal{X}^A(\vec{\theta})$ and $X^A(\vec{x})$ determines a local Lie group up to (local analytic) isomorphism (i.e. up to a linear transformation in the Lie algebra).

- Converse of the 3rd theorem:

  An abstract Lie algebra (see Section 5.4) determines a simply connected Lie group uniquely up to isomorphism.

  Extension: For each given finite-dimensional Lie algebra $\mathcal{L}$ there is even a matrix Lie group with $\mathcal{L}$ as Lie algebra.

Implications:

- All simply connected Lie groups (universal covering groups) can be classified by classifying Lie algebras.
  The classification of matrix Lie algebras provides also a classification of all abstract Lie algebras.

- All Lie groups for a given Lie algebra can be obtained from the corresponding universal covering group G by determining the discrete, invariant subgroups G_d of G and deducing the factor groups G/G_d.

  Note: Since G is simply connected, the subgroups G_d consist of elements that commute with all $g \in G$, i.e. the G_d are the subgroups of the centre of G.

Special case: matrix Lie groups

Matrix transformation:

$$\vec{x}\,' = \vec{F}(\vec{\theta}, \vec{x}) = U(\vec{\theta})\,\vec{x}. \tag{5.16}$$

Construction of generators:

$$\vec{u}^A(\vec{x}) = \frac{\partial U(\vec{\theta})}{\partial \theta_A}\bigg|_{\vec{\theta}=\vec{0}}\,\vec{x} \equiv -\mathrm{i}T^A\,\vec{x}, \qquad T^A = N \times N \text{ matrix}. \tag{5.17}$$

$\hookrightarrow$ Generators for transformation (5.16):

- as differential operators:

$$X^A(\vec{x}) = -\mathrm{i}u^A_a(\vec{x})\,\frac{\partial}{\partial x_a} = -T^A_{ab}\,x_b\,\frac{\partial}{\partial x_a}; \tag{5.18}$$

- as matrices: The T^A obey the Lie commutators:

$$\begin{aligned}
[X^A(\vec{x}), X^B(\vec{x})] &= \left[T^A_{ab}\,x_b\,\frac{\partial}{\partial x_a}, T^B_{cd}\,x_d\,\frac{\partial}{\partial x_c}\right] = T^A_{ab}\,T^B_{cd}\,\underbrace{\left[x_b\,\frac{\partial}{\partial x_a}, x_d\,\frac{\partial}{\partial x_c}\right]}_{=x_b\delta_{ad}\partial_c - x_d\delta_{cb}\partial_a} \\
&= (T^B\,T^A)_{cb}\,x_b\,\frac{\partial}{\partial x_c} - (T^A\,T^B)_{ad}\,x_d\,\frac{\partial}{\partial x_a} = -[T^A\,T^B]_{ab}\,x_b\,\frac{\partial}{\partial x_a} \\
&= \mathrm{i}f^{AB}{}_C\,X^C(\vec{x}) = -\mathrm{i}f^{AB}{}_C\,T^C_{ab}\,x_b\,\frac{\partial}{\partial x_a}.
\end{aligned}$$

$$\Rightarrow\ [T^A, T^B] = \mathrm{i}f^{AB}{}_C\,T^C. \tag{5.19}$$

5.2 One-parameter subgroups, exponentiation, and BCH formula

Problem: Functions $\theta''_A = f_A(\vec{\theta}\,', \vec{\theta}\,)$ in general hard to get, but

- one-parameter subgroups admit canonical form $\theta'' = \theta' + \theta$;
- general case ruled by Baker–Campbell–Hausdorff (BCH) formula.

Theorem on one-parameter subgroups

Each direction in group-parameter space of a Lie group G, defined by some unit vector $\vec{n} = (n_A)$, determines a one-parameter subgroup $G_{\vec{n}}$ with the multiplication property $g(\lambda' + \lambda) = g(\lambda')\, g(\lambda)$, where $g(\lambda) \equiv g(\vec{\theta} = \lambda\vec{n})$.
The corresponding Lie group transformation on some vector $\vec{x} \in \mathbb{R}^N$ is given by

$$\vec{x}(\lambda) = \mathcal{U}(\lambda)\, \vec{x}, \qquad \mathcal{U}(\lambda) \equiv \exp\left\{ \mathrm{i}\lambda n_A X^A(\vec{x}) \right\}, \tag{5.20}$$

with the generators $X^A(\vec{x})$ of G at the start point $\vec{x}(0) = \vec{x}$ of the trajectory:

$$X^A(\vec{x}) = -\mathrm{i} u^A_a(\vec{x})\, \frac{\partial}{\partial x_a}. \tag{5.21}$$

Proof:
Subgroup defined by constructing a trajectory $\vec{x}(\lambda)$ with $\vec{x}(0) = \vec{x}$ which corresponds to some Lie group transformation with $\vec{\theta} = \lambda\vec{n}$:

- Lie's 1st theorem for one-parameter group $G_{\vec{n}}$:

$$\frac{\mathrm{d}x_a(\lambda)}{\mathrm{d}\lambda} = n_A u^A_a\big(\vec{x}(\lambda)\big), \qquad \vec{x}\,' = \vec{x}(\lambda), \tag{5.22}$$

  where $\Theta(\vec{\theta}) = \Psi(\vec{\theta}) = 1$, since $\lambda'' \overset{!}{=} \lambda' + \lambda$.

- As 1st-order ordinary differential equation, (5.22) has a unique solution for given $\vec{x}(0) = \vec{x}$.
  $\hookrightarrow$ Check that (5.20) solves (5.22): $\vec{x}(0) = \vec{x}$ is obvious.

$$\frac{\mathrm{d}\vec{x}(\lambda)}{\mathrm{d}\lambda} = \frac{\mathrm{d}\mathcal{U}(\lambda)}{\mathrm{d}\lambda}\, \vec{x} = \mathcal{U}(\lambda)\, \mathrm{i} n_A X^A(\vec{x})\, \vec{x} = \mathcal{U}(\lambda)\, n_A \vec{u}^A(\vec{x}). \tag{5.23}$$

  $\Rightarrow$ Still to show:

$$\mathcal{U}(\lambda)\, n_A \vec{u}^A(\vec{x}) = n_A \vec{u}^A\big(\vec{x}(\lambda)\big). \tag{5.24}$$

- Proof of (5.24) with auxiliary relation for linear operators A, B: (Exercise!)

$$\exp(A)\, B\, \exp(-A) = \exp(\mathrm{ad}_A)(B), \qquad (\mathrm{ad}_A)^k(B) \equiv \underbrace{[A, [\ldots, [A, B], \ldots]]}_{k \text{ commutators}}. \tag{5.25}$$

Choose $A = \mathrm{i}\lambda n_A X^A(\vec{x})$ and $B = x_b$:

$$\mathrm{ad}_A(B) = [A, B] = \mathrm{i}\lambda n_A \vec{u}^A(\vec{x}) \left(\frac{\partial}{\partial \vec{x}}\, x_b\right) = \text{function of } \vec{x} \text{ (multiplicative op.)},$$

$$(\mathrm{ad}_A)^k(B) = \left(\left(\mathrm{i}\lambda n_A \vec{u}^A(\vec{x}) \frac{\partial}{\partial \vec{x}}\right)^k x_b\right).$$

$$\hookrightarrow\ \exp(\mathrm{ad}_A)(B) = \mathcal{U}(\lambda)\, x_b = x_b(\lambda)$$
$$= \exp(A)\, B\, \exp(-A) = \mathcal{U}(\lambda)\, x_b\, \mathcal{U}(\lambda)^{-1}. \tag{5.26}$$

Since $\vec{u}^A(\vec{x})$ is analytic, $\mathcal{U}(\lambda)\, x_b\, \mathcal{U}(\lambda)^{-1} = x_b(\lambda)$ implies (5.24):

$$\mathcal{U}(\lambda)\, n_A \vec{u}^A(\vec{x}) = \mathcal{U}(\lambda)\, n_A \vec{u}^A(\vec{x})\, \mathcal{U}(\lambda)^{-1} \cdot 1 = n_A \vec{u}^A\left(\mathcal{U}(\lambda)\, \vec{x}\, \mathcal{U}(\lambda)^{-1}\right) \cdot 1$$
$$= n_A \vec{u}^A\big(\vec{x}(\lambda)\big).$$

#

Special case: matrix Lie groups

$$\vec{x}\,' = \vec{F}(\vec{\theta}, \vec{x}) = U(\vec{\theta})\, \vec{x}. \tag{5.27}$$

Transformation operator for one-parameter Lie group: $\vec{\theta} = \lambda\, \vec{n}$.

$$\mathcal{U}(\lambda) = \exp\left\{\mathrm{i}\lambda n_A X^A(\vec{x})\right\} = \exp\left\{-\mathrm{i}\lambda n_A T^A_{ab}\, x_b\, \frac{\partial}{\partial x_a}\right\}. \tag{5.28}$$

$\hookrightarrow$ Derivation of matrix transformation $U(\vec{\theta}) = U(\lambda \vec{n})$: $(\vec{x} = x_a \vec{e}_a)$

$$-\mathrm{i}\theta_A T^A_{ab}\, x_b\, \frac{\partial}{\partial x_a}\, \vec{x} = -\mathrm{i}\theta_A T^A_{ab}\, x_b\, \vec{e}_a = -\mathrm{i}\theta_A T^A\, \vec{x},$$

$$\left(-\mathrm{i}\theta_A T^A_{ab}\, x_b\, \frac{\partial}{\partial x_a}\right)^k \vec{x} = \left(-\mathrm{i}\theta_A T^A\right)^k \vec{x}.$$

$$\Rightarrow\ \mathcal{U}(\lambda)\, \vec{x} = \exp\left\{-\mathrm{i}\theta_A T^A\right\}\, \vec{x}. \qquad \Rightarrow\ U(\vec{\theta}) = \exp\left\{-\mathrm{i}\theta_A T^A\right\}. \tag{5.29}$$

Convergence and consistency of exp

- The exponential form of the transformations $\mathcal{U}(\lambda)$ and $U(\vec{\theta})$ always converge.
- In the identity component of compact groups, all group transformations can be written in exponential form. For non-compact groups, in general a product of a finite number of exponentials is required.

Non-canonical parametrizations of group elements

The canonical form of matrix Lie group elements

$$U(\vec{\theta}) = \exp\left\{-\mathrm{i}\theta_A T^A\right\} \tag{5.30}$$

is sometimes inconvenient to calculate matrix elements $\langle\psi|\, U(\vec{\theta})\, |\phi\rangle$!

$\hookrightarrow$ Often non-canonical forms like

$$U(\alpha_1, \alpha_2, \dots) = \exp\{-\mathrm{i}\alpha_1 \tilde{T}^1\}\, \exp\{-\mathrm{i}\alpha_2 \tilde{T}^2\}\, \dots \tag{5.31}$$

are more convenient if some of the new generators $\tilde{T}^A$ are

- diagonal (exp easy to compute) or
- nilpotent (exp series truncates).

Example: Euler-angle parametrizations of SO(3) and SU(2) elements:

$$D(\vec{\theta}) = \exp\big\{-\mathrm{i}\vec{\theta}\,\vec{J}\big\} = D(\alpha,\beta,\gamma) = \exp\big\{-\mathrm{i}\alpha\, J_3\big\}\, \exp\big\{-\mathrm{i}\beta\, J_2\big\}\, \exp\big\{-\mathrm{i}\gamma\, J_3\big\},$$

with J_3 = diagonal in the usual representations.

Baker–Campbell–Hausdorff (BCH) formula

Given two elements X, Y in the Lie algebra $\mathcal{L}$ of a Lie group G sufficiently close to 0, the following relation holds:

$$-\mathrm{i}\ln\left(\mathrm{e}^{\mathrm{i}X}\,\mathrm{e}^{\mathrm{i}Y}\right) = X + \int_0^1 \mathrm{d}t\, g\left(e^{\mathrm{i}\,\mathrm{ad}_X}\, e^{\mathrm{i}t\,\mathrm{ad}_Y}\right)(Y) \;\in \mathcal{L}, \tag{5.32}$$

with

$$g(z) \equiv \frac{\ln z}{1 - 1/z} = \text{analytic function for } |z-1| < 1. \tag{5.33}$$

$\Rightarrow$ BCH formula explicitly constructs the group element $\mathrm{e}^{\mathrm{i}Z} = \mathrm{e}^{\mathrm{i}X}\,\mathrm{e}^{\mathrm{i}Y}$ for given X, Y.

Differential form:

$$\ln\left(\mathrm{e}^{\mathrm{i}X}\,\mathrm{e}^{\mathrm{i}Y}\right) = \mathrm{i}X + \mathrm{i}Y - \frac{1}{2}[X,Y] - \frac{\mathrm{i}}{12}[X,[X,Y]] + \frac{\mathrm{i}}{12}[Y,[X,Y]] + \dots, \tag{5.34}$$

where $\dots$ stands for multiple commutators with at least 4 operators X, Y.

$\hookrightarrow$ Form useful to obtain local information on functions $f_A(\vec{\theta}\,', \vec{\theta})$ for small $\vec{\theta}\,'$, $\vec{\theta}$.

Comments:

- BCH formula and its proof rather non-trivial (see, e.g., [6]).
- Special case: (proven in Exercise 1.4)

$$\mathrm{e}^{\mathrm{i}X}\,\mathrm{e}^{\mathrm{i}Y} = \mathrm{e}^{\mathrm{i}X + \mathrm{i}Y - \frac{1}{2}[X,Y]} \quad \text{if } [X,[X,Y]] = [Y,[X,Y]] = 0. \tag{5.35}$$

5.3 Invariant group integration

Aim: generalization of $\sum_g F(g) = \sum_g F(g'g)\ \ \forall g' \in G$ (= finite group), which

- is valid due to the rearrangement lemma,
- attributes equal weight (=1) to each element $g \in G$,

to Lie group with elements $g = g(\vec{\theta}\,)$:

$$\sum_g F(g) \ \rightarrow\ \int_G \mathrm{d}\mu_g\, F(g) = \int \mathrm{d}^n\vec{\theta}\ \underbrace{\rho(\vec{\theta}\,)}_{\text{density function}}\ F(g(\vec{\theta}\,)). \tag{5.36}$$

$\hookrightarrow$ "Left invariance" requirement: $\underbrace{\mathrm{d}\mu_g}_{\text{volume element at } g} = \underbrace{\mathrm{d}\mu_{g'g}}_{\text{volume element at } g'g}\ \ \forall g' \in G.$

Construction of $\rho(\vec{\theta}\,)$:

$$\begin{aligned} &g'' = g'g, \qquad g(\vec{\theta}\,'') = g(\vec{\theta}\,')\, g(\vec{\theta}\,), \\ &\theta''_A = f_A(\vec{\theta}\,', \vec{\theta}\,), \quad \theta_A = f_A(0, \vec{\theta}\,) = f_A(\vec{\theta}\,, 0), \quad \text{since } g(\vec{0}\,) = e. \end{aligned} \tag{5.37}$$

Taking $\vec{\theta} \rightarrow \hat{\vec{\theta}}$ = infinitesimal yields

$$\underbrace{\mathrm{d}^n\vec{\theta}\,'}_{\substack{\text{volume element left} \\ \text{translated from } \vec{0} \text{ to } \vec{\theta}\,'}} = \underbrace{\mathrm{d}^n\hat{\vec{\theta}}}_{\substack{\text{volume} \\ \text{element at } \vec{0}}}\ \det\left(\frac{\partial f_A(\vec{\theta}\,', \vec{\theta}\,)}{\partial \theta_B}\right)\Bigg|_{\vec{\theta}=\vec{0}} \equiv \mathrm{d}^n\hat{\vec{\theta}}\, J(\vec{\theta}\,'). \tag{5.38}$$

$\Rightarrow$ Definition:

$$\rho(\vec{\theta}\,) \equiv \frac{\rho(\vec{0}\,)}{J(\vec{\theta}\,)}, \qquad \rho(\vec{0}\,) = \text{convention}. \tag{5.39}$$

Check invariance of $\mathrm{d}\mu_g$:

$$\mathrm{d}\mu_{g'g} = \mathrm{d}^n\vec{\theta}\,'\, \rho(\vec{\theta}\,') = \mathrm{d}^n\hat{\vec{\theta}}\, \rho(\vec{0}\,) = \mathrm{d}^n\vec{\theta}\ \rho(\vec{\theta}\,) = \mathrm{d}\mu_g. \tag{5.40}$$

Theorem for compact groups:

a) $\int \mathrm{d}\mu_g = V_G < \infty$ exists ("Haar measure"), usual convention: $V_G = 1$.
Fixing V_G, the Haar measure is unique.

b) The "left-invariant" measure $\mathrm{d}\mu_g$ is also "right invariant", i.e.

$$\int_G \mathrm{d}\mu_g\, F(g) = \int_G \mathrm{d}\mu_g\, F(g'g) = \int_G \mathrm{d}\mu_g\, F(gg') \quad \forall g' \in G. \tag{5.41}$$

For a proof of a), see math. literature.

A sketchy proof of b) in 3 steps:

1. Show that $\mathrm{d}\mu_{\hat{g}} = \mathrm{d}\mu_{g'\hat{g}g'^{-1}}$ for infinitesimal $\hat{g}$, i.e. $\hat{g} = g(\delta\hat{\vec{\theta}})$, $\delta\hat{\vec{\theta}} = \text{inf.}$

   $\tilde{g} = g'\hat{g}g'^{-1} = \text{inf.}$ with $\tilde{g} = g(\delta\tilde{\vec{\theta}})$ and $\delta\tilde{\vec{\theta}} = M\,\delta\hat{\vec{\theta}}$ with some matrix M.

   $\hookrightarrow$ Consider $\tilde{g}^{(m)} = (g')^m\hat{g}(g'^{-1})^m = g(\delta\tilde{\vec{\theta}}^{(m)})$, where $\delta\tilde{\vec{\theta}}^{(m)} = M^m\,\delta\hat{\vec{\theta}}$:
   If G is compact, there are two possibilities:
   (i) $\tilde{g}$ has finite order N, then $M^N = \mathbb{1}$.
   (ii) $\tilde{g}$ has infinite order, then $\lim\limits_{m\to\infty}\tilde{g}^{(m)} = \tilde{g}^{(\infty)}$ and thus M^∞ have to exist.

   $\Rightarrow$ In either case $\det M = 1$ and thus $\mathrm{d}^n\tilde{\vec{\theta}} = \mathrm{d}^n\hat{\vec{\theta}}$, so that

$$\mathrm{d}\mu_{\hat{g}} = \mathrm{d}^n\hat{\vec{\theta}}\,\rho(\vec{0}\,) = \mathrm{d}^n\tilde{\vec{\theta}}\,\rho(\vec{0}\,) = \mathrm{d}\mu_{\tilde{g}} = \mathrm{d}\mu_{g'\hat{g}g'^{-1}} \quad \forall g' \in G. \tag{5.42}$$

2. Generalization of $\mathrm{d}\mu_g = \mathrm{d}\mu_{g'gg'^{-1}}$ to any g:
   Let $\hat{g}$ be inf. and $g = \bar{g}\hat{g}$, then using (5.42) for $\hat{g}$ and left invariance of $\mathrm{d}\mu_g$:

$$\mathrm{d}\mu_g = \mathrm{d}\mu_{\bar{g}\hat{g}} = \mathrm{d}\mu_{\hat{g}} \underset{(5.42)}{=} \mathrm{d}\mu_{g'\hat{g}g'^{-1}} = \mathrm{d}\mu_{\hat{g}g'^{-1}} = \mathrm{d}\mu_{\bar{g}\hat{g}g'^{-1}} = \mathrm{d}\mu_{gg'^{-1}} = \mathrm{d}\mu_{g'gg'^{-1}}. \tag{5.43}$$

3. Proof of right invariance of $\mathrm{d}\mu_g$: $\quad \mathrm{d}\mu_{gg'} = \mathrm{d}\mu_{g'^{-1}gg'} \underset{(5.43)}{=} \mathrm{d}\mu_g$. #

Example: Haar measures of SU(2) ans SO(3)

A suitable parametrization of SU(2) matrices:

$$U(\vec{x}) = x_0\,\mathbb{1} - \mathrm{i}\vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x_0 - \mathrm{i}x_3 & -\mathrm{i}x_1 - x_2 \\ -\mathrm{i}x_1 + x_2 & x_0 + \mathrm{i}x_3 \end{pmatrix}, \quad x_0 = \pm\sqrt{1-\vec{x}^{\,2}}. \tag{5.44}$$

Relation to the form (3.18) with "rotation vector" $\vec{\theta} = \theta\vec{e}$ ($\vec{e}^{\,2} = 1$):

$$x_0 = \cos\tfrac{\theta}{2}, \qquad \vec{x} = \sin\tfrac{\theta}{2}\,\vec{e}. \tag{5.45}$$

Variations of U before and after translation to $U(\vec{x})$:

$$U(\delta\vec{x}) = \begin{pmatrix} -\mathrm{i}\delta x_3 & -\mathrm{i}\delta x_1 - \delta x_2 \\ -\mathrm{i}\delta x_1 + \delta x_2 & \mathrm{i}\delta x_3 \end{pmatrix},$$
$$U(\vec{x}\,' + \delta\vec{x}\,') = \begin{pmatrix} x_0' - \mathrm{i}x_3' + \delta x_0' - \mathrm{i}\delta x_3' & -\mathrm{i}x_1' - x_2' - \mathrm{i}\delta x_1' - \delta x_2' \\ -\mathrm{i}x_1' + x_2' - \mathrm{i}\delta x_1' + \delta x_2' & x_0' + \mathrm{i}x_3' + \delta x_0' + \mathrm{i}\delta x_3' \end{pmatrix}, \quad \delta x_0' = -x_n'\delta x_n'. \tag{5.46}$$

Transformation of differentials and volume element from $U(\vec{x}\,' + \delta\vec{x}\,') = U(\vec{x}\,')\,U(\delta\vec{x})$:

$$\delta\vec{x} = \begin{pmatrix} x_0' & -x_3' & x_2' \\ x_3' & x_0' & -x_1' \\ -x_2' & x_1' & x_0' \end{pmatrix}\delta\vec{x}\,' \quad \Rightarrow \quad \mathrm{d}^3\vec{x} = |x_0(x_0^2 + x_nx_n)|\,\mathrm{d}^3\vec{x}\,' = \underbrace{\sqrt{1-\vec{x}\,'^2}}_{=J(\vec{x}\,')}\,\mathrm{d}^3\vec{x}\,'. \tag{5.47}$$

⇒ Haar measure of SU(2):

$$\int_{\mathrm{SU}(2)} \mathrm{d}\mu_U = \frac{1}{2\pi^2}\int_{|\vec{x}|\le 1}\frac{\mathrm{d}^3\vec{x}}{\sqrt{1-\vec{x}^{\,2}}}\sum_{x_0=\pm\sqrt{1-\vec{x}^{\,2}}} = \frac{1}{\pi^2}\int \mathrm{d}^4x\,\delta(1-x_0^2-\vec{x}^{\,2})$$

$$= \frac{1}{8\pi^2}\int \mathrm{d}\Omega \int_0^{2\pi}\mathrm{d}\theta\,(1-\cos\theta\,), \qquad \Omega = \text{solid angle of } \vec{e}. \tag{5.48}$$

⇒ Haar measure of SO(3): (only $x_0 = +\sqrt{1-\vec{x}^{\,2}}$, i.e. $0\le\theta\le\pi$)

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mu_U = \frac{1}{\pi^2}\int_{|\vec{x}|\le 1}\frac{\mathrm{d}^3\vec{x}}{\sqrt{1-\vec{x}^{\,2}}}\bigg|_{x_0=\sqrt{1-\vec{x}^{\,2}}} = \frac{1}{4\pi^2}\int \mathrm{d}\Omega\int_0^{\pi}\mathrm{d}\theta\,(1-\cos\theta\,). \tag{5.49}$$

Reparametrization in terms of Euler angles:

$$x_1 = \sin\tfrac{\beta}{2}\sin\phi, \quad x_2 = \sin\tfrac{\beta}{2}\cos\phi, \quad x_3 = \sin\tfrac{\beta}{2}\sin\chi, \quad x_0 = \cos\tfrac{\beta}{2}\cos\chi, \tag{5.50}$$

$$\left.\begin{array}{l} 0\le\phi=\tfrac{1}{2}(\gamma-\alpha)\le 2\pi \\ 0\le\chi=\tfrac{1}{2}(\gamma+\alpha)\le 2\pi \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} 0\le\alpha\le 2\pi, \\ 0\le\gamma\le 4\pi, \end{array}\right.$$

$$0\le\beta\le\pi, \quad x_0<0 \text{ included.} \tag{5.51}$$

$$\frac{\mathrm{d}^3\vec{x}}{\sqrt{1-\vec{x}^{\,2}}} = \frac{\mathrm{d}\phi\,\mathrm{d}\sin\frac{\beta}{2}\,\sin\frac{\beta}{2}\,\mathrm{d}\sin\chi\,\cos\frac{\beta}{2}}{\cos\frac{\beta}{2}\cos\chi} = \mathrm{d}\phi\,\mathrm{d}\sin\tfrac{\beta}{2}\,\sin\tfrac{\beta}{2}\,\mathrm{d}\chi = \tfrac{1}{8}\mathrm{d}\alpha\,\mathrm{d}\cos\beta\,\mathrm{d}\gamma.$$

$$\Rightarrow \int_{\mathrm{SU}(2)}\mathrm{d}\mu_U = \frac{1}{16\pi^2}\int_0^{2\pi}\mathrm{d}\alpha\int_{-1}^{1}\mathrm{d}\cos\beta\int_0^{4\pi}\mathrm{d}\gamma, \tag{5.52}$$

$$\int_{\mathrm{SO}(3)}\mathrm{d}\mu_U = \frac{1}{8\pi^2}\int_0^{2\pi}\mathrm{d}\alpha\int_{-1}^{1}\mathrm{d}\cos\beta\int_0^{2\pi}\mathrm{d}\gamma. \tag{5.53}$$

Implications for compact groups: (similarity to finite groups!)

- All finite-dimimensional representations can be taken unitary, and all irreducible representations are finite dimensional.
- Orthogonality relations of (unitary) irreducible representations $D^{(j)}$:

$$\int_G \mathrm{d}\mu_g\, D^{(j)}_{ab}(g)^*\, D^{(k)}_{cd}(g) = \delta_{jk}\,\delta_{ac}\,\delta_{bd}\,\frac{V_G}{n_j}, \qquad n_j = \dim D^{(j)}. \tag{5.54}$$

- Completeness relation ("Peter-Weyl theorem"):

$$\sum_j n_j\,\mathrm{Tr}\left\{D^{(j)}(g)^\dagger\, D^{(j)}(g')\right\} = \delta(g-g') \equiv \frac{\delta(\vec{\theta}-\vec{\theta}\,')}{\rho(\vec{\theta}\,)}, \tag{5.55}$$

where $\sum_j n_j$ runs over all inequivalent irreducible unitary representations.

⇒ Any (square-integrable) function $F(g)$ on G can be expanded:

$$F(g) = \sum_{j,a,b} f^{(j)}_{ab}\, D^{(j)}_{ab}(g), \qquad f^{(j)}_{ab} = \frac{n_j}{V_G}\int_G \mathrm{d}\mu_g\, F(g)\, D^{(j)}_{ab}(g)^*. \tag{5.56}$$

5.4 Lie algebras

Definitions: (more abstract algebraic versions)

- "Algebra" $\mathcal{A} \equiv$ vector space with a bilinear product operation:

$$a, b \in \mathcal{A} \quad \Rightarrow \quad a \circ b \in \mathcal{A}, \tag{5.57}$$

$$a, \dots, d \in \mathcal{A};\ \ \alpha, \dots, \delta \in \mathbb{K} = \mathbb{R}, \mathbb{C}$$
$$\Rightarrow \quad (\alpha a + \beta b) \circ (\gamma c + \delta d) = \alpha\gamma(a \circ b) + \alpha\delta(a \circ d) + \beta\gamma(b \circ c) + \beta\delta(b \circ d). \tag{5.58}$$

- "Lie algebra" $\mathcal{L} \equiv$ finite-dimensional algebra with a "Lie product" $[.,.]$ as product operation:

$$[x, x] = 0 \ \ \forall x \in \mathcal{L} \quad \Rightarrow \quad [x, y] = -[y, x] \ \ \forall x, y \in \mathcal{L}, \tag{5.59}$$

$$\text{Jacobi identity:} \quad [x, [y, z]] + \text{cyclic} = 0 \ \ \forall x, y, z \in \mathcal{L}. \tag{5.60}$$

$d_{\mathcal{L}} = \dim \mathcal{L} \equiv$ dimension of $\mathcal{L}$ as vector space.

Example: $[x, y] = x \circ y - y \circ x$ for an associative product $\circ$.

In a given basis $\{T^A\}_{A=1}^{\dim \mathcal{L}}$ of $\mathcal{L}$, each $x \in \mathcal{L}$ can be written as $x = x_A T^A$, and the closure of $\mathcal{L}$ under $[.,.]$ implies

$$[T^A, T^B] \equiv \mathrm{i} f^{AB}{}_C T^C, \qquad f^{AB}{}_C = -f^{BA}{}_C, \tag{5.61}$$

and the Jacobi identity implies $f^{AB}{}_C f^{DC}{}_E + \text{cyclic} = 0$.

- A "complexification" $\mathcal{L}_{\mathbb{C}}$ of a real Lie algebra $\mathcal{L}$ is spanned by complex linear combinations of a basis of generators $\{T^A\}$ of $\mathcal{L}$.

Adjoint representation and Killing form:

- "Adjoint representation" $(T^A_{\mathrm{ad}})^B{}_C \equiv -\mathrm{i} f^{AB}{}_C$, $\quad \mathrm{ad}_x = x_A T^A_{\mathrm{ad}}$.
$\hookrightarrow$ $[T^A_{\mathrm{ad}}, T^B_{\mathrm{ad}}] = \mathrm{i} f^{AB}{}_C T^C_{\mathrm{ad}}$ by Jacobi identity.

Note: $\{\mathrm{ad}_x\}$ provide a representation with $\mathcal{L}$ as representation space itself:

$$\mathrm{ad}_x(y) = [x, y], \tag{5.62}$$

$$\mathrm{ad}_{[x,y]}(z) = [\mathrm{ad}_x, \mathrm{ad}_y](z). \tag{5.63}$$

- "Cartan–Killing form" g:

$$g^{AB} \equiv \mathrm{Tr}\left(T^A_{\mathrm{ad}} T^B_{\mathrm{ad}}\right) = -f^{AC}{}_D f^{BD}{}_C = g^{BA}. \tag{5.64}$$

Notation:

$$(x, y) \equiv \mathrm{Tr}\,(\mathrm{ad}_x, \mathrm{ad}_y) = x_A y_B \, \mathrm{Tr}\left(T^A_{\mathrm{ad}} T^B_{\mathrm{ad}}\right) = x_A y_B g^{AB}. \tag{5.65}$$

- $\mathcal{L}$ decomposes into a "direct sum" of two Lie algebras, $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, if $[x_1, x_2] = 0 \ \forall x_1 \in \mathcal{L}_1, x_2 \in \mathcal{L}_2$. This implies:

$$f^{AB}{}_C = 0 \quad \text{if} \quad T^A \in \mathcal{L}_1, T^B \in \mathcal{L}_2 \ \text{ or vice versa}, \tag{5.66}$$

$$(x_1, x_2) = 0 \quad \text{if} \quad x_1 \in \mathcal{L}_1, x_2 \in \mathcal{L}_2. \tag{5.67}$$

Extension of some group properties to Lie algebras:

- "Invariant Lie subalgebra" (="ideal") $\mathcal{H} \equiv$ subalgebra with $[\mathcal{H}, \mathcal{L}] \subseteq \mathcal{H}$.
- "Simple Lie algebra" $\equiv$ Lie algebra with dim > 1 without a proper ideal (i.e. $\neq \{0\}, \mathcal{L}$).
- "Semisimple Lie algebra" $\equiv$ Lie algebra with dim > 1 without a proper abelian ideal.
- "Compact Lie algebra" $\equiv$ real Lie algebra corresponding to a compact Lie group G.
  - G = compact. $\Rightarrow$ finite-dim. representations can be chosen unitary: $u = \exp\left\{\mathrm{i}\theta_A T^A\right\}$ = unitary. $u^\dagger = u^{-1} \quad \Rightarrow \quad \left(T^A\right)^\dagger = T^A$ = hermitian.
  - $\left(T^A_{\mathrm{ad}}\right)^\dagger = T^A_{\mathrm{ad}} \quad \Rightarrow \quad f^{AB}{}_C$ = real and $f^{AB}{}_C = -f^{AC}{}_B$.

Some facts about (semi)simplicity: (some proofs beyond the scope of this lecture)

a) $\mathcal{L}$ = semisimple $\quad \Leftrightarrow \quad (g^{AB})$ = non-singular. ("Cartan's criterion")

$\Rightarrow$ Define inverse of g: $\quad g^{AB} g_{BC} = \delta^A_C$.

$\hookrightarrow$ g acts as metric to raise/lower indices:

$$
\begin{aligned}
f^{ABC} &\equiv f^{AB}{}_D g^{DC} = -f^{AB}{}_D f^{CE}{}_F f^{DF}{}_E \\
&= \left(f^{BF}{}_D f^{DA}{}_E + f^{FA}{}_D f^{DB}{}_E\right) f^{CE}{}_F \qquad \text{(Jacobi id.)} \\
&= -f^{BF}{}_D f^{AD}{}_E f^{CE}{}_F + f^{AF}{}_D f^{BD}{}_E f^{CE}{}_F \\
&= \mathrm{i}\,\mathrm{Tr}\left(T^B_{\mathrm{ad}} T^A_{\mathrm{ad}} T^C_{\mathrm{ad}} - T^A_{\mathrm{ad}} T^B_{\mathrm{ad}} T^C_{\mathrm{ad}}\right) \\
&= \text{antisymmetric in } A, B, C.
\end{aligned}
\tag{5.68}
$$

$$
\begin{aligned}
\Rightarrow \quad ([x,y],z) &= \mathrm{Tr}\left(T^A_{\mathrm{ad}} T^B_{\mathrm{ad}} T^C_{\mathrm{ad}} - T^B_{\mathrm{ad}} T^A_{\mathrm{ad}} T^C_{\mathrm{ad}}\right) x_A y_B z_C = \mathrm{i} f^{ABC} x_A y_B z_C \\
&= ([y,z],x) = ([z,x],y) \\
&= (x,[y,z]) = \dots
\end{aligned}
\tag{5.69}
$$

b) $\mathcal{L}$ = semisimple & compact $\quad \Leftrightarrow \quad (g^{AB})$ = positive definite.

Proof of "$\Rightarrow$":

Use compactness: $\quad g^{AB} = -f^{AC}{}_D f^{BD}{}_C = +f^{AC}{}_D f^{BC}{}_D$.

$\hookrightarrow \quad (x,x) = x_A x_B g^{AB} = \left(x_A f^{AC}{}_D\right)\left(x_B f^{BC}{}_D\right) = \left(x_A f^{AC}{}_D\right)^2 \geq 0.$

But: $\quad (x,x) > 0$ for $x \neq 0$ due to semisimplicity of $\mathcal{L}$, see a). #

c) Every complex semisimple Lie algebra can be obtained as complexification of a (real!) compact semisimple Lie algebra.

d) $\mathcal{L}$ = simple $\Rightarrow$ adjoint representation is faithful (=isomorphic to $\mathcal{L}$).

Proof:

$$\begin{aligned}\ker(\mathrm{ad}_x) &= \{x \in \mathcal{L} \mid \mathrm{ad}_x(y) = [x,y] = 0\ \forall y \in \mathcal{L},\ \text{i.e. } \mathrm{ad}_x = 0\} \\ &= \text{centre of } \mathcal{L}\ (= \text{set of commuting elements})\end{aligned}$$

$\hookrightarrow$ defines an ideal $\mathcal{I}$ of $\mathcal{L}$.

But: $\mathcal{L}$ = simple $\Rightarrow \mathcal{I} = \{0\}$ or $\underbrace{\mathcal{L}}_{\text{impossible, otherwise } \mathcal{L} = \text{abelian}}$

$\Rightarrow \ker(\mathrm{ad}_x) = \{0\}$. #

e) $\mathcal{L}$ = semisimple $\Leftrightarrow$ $\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ with $\mathcal{L}_k$ = simple and $[\mathcal{L}_k, \mathcal{L}] = \mathcal{L}_k$ (=ideal).

Proof: based on a).

"$\Rightarrow$":
- Be $\mathcal{I}$ an ideal of $\mathcal{L}$ (if there is none, there is nothing to show).
  $\hookrightarrow [\mathcal{I}, \mathcal{L}] \subseteq \mathcal{I}$.
- Def.: $C \equiv$ complement of $\mathcal{I}$ w.r.t. g, i.e. $(C, \mathcal{I}) = 0$.

$$\left.\begin{aligned} \Rightarrow \big([C,\mathcal{I}],\mathcal{I}\big) &\underset{a)}{=} \big(\underbrace{[\mathcal{I},\mathcal{I}]}_{\subseteq \mathcal{I}}, C\big) = 0 \\ \big(\underbrace{[C,\mathcal{I}]}_{\subseteq \mathcal{I},\ \text{ideal!}}, C\big) &= 0 \end{aligned}\right\} \Rightarrow [C,\mathcal{I}] = \{0\},\ \text{since } g = \text{non-singular.}$$

  $\Rightarrow \mathcal{L} = \mathcal{I} \oplus C$.
- $\mathcal{I}$ and C are semisimple, since the restrictions of g on $\mathcal{I}$ or C are non-singular:

  $x \in \mathcal{L},\ x = x_\mathcal{I} + x_C,\ x_\mathcal{I} \in \mathcal{I},\ x_C \in C$ $\quad y$ analogously.

  $\hookrightarrow (x,y) = (x_\mathcal{I}, y_\mathcal{I}) + (x_C, y_C)$.
- Repeat decomposition of $\mathcal{I}$ and C recursively until only simple subalgebras remain.

"$\Leftarrow$": $\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n, \quad [\mathcal{L}_k, \mathcal{L}_l] = 0 \quad \text{for } k \neq l.$

Let $x = \sum_{k=1}^{n} x_k,\ x_k \in \mathcal{L}_k, \quad y$ = analogously.

$$\hookrightarrow (x,y) = \sum_{k=1}^{n} \underbrace{(x_k, y_k)}_{= \text{non-singular metric on } \mathcal{L}_k,\ \text{since } \mathcal{L}_k \text{ is simple.}} = \text{non-singular.} \quad \Rightarrow \mathcal{L} = \text{semi-simple.}$$

Recall: If $T^A \in \mathcal{L}_k$, then ${f^{AB}}_C = 0$ if $T^B \notin \mathcal{L}_k$.

$\Rightarrow g^{AB}\big|_{\mathcal{L}_k}$ yields metric on $\mathcal{L}_k$. #

f) $\mathcal{L}$ = semisimple $\quad\Leftrightarrow\quad$ $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$,
i.e. each element can be written as commutator.

Proof of "⇒": based on previous property e).

$$\mathcal{L} = \text{semisimple} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n, \quad \mathcal{L}_k = \text{simple} = \text{ideal}, \quad [\mathcal{L}_k, \mathcal{L}_l] = 0 \text{ for } k \neq l.$$

$$\hookrightarrow \quad [\mathcal{L}, \mathcal{L}] = \underbrace{[\mathcal{L}_1, \mathcal{L}_1]}_{=\mathcal{L}_1} \oplus \cdots \oplus \underbrace{[\mathcal{L}_n, \mathcal{L}_n]}_{=\mathcal{L}_n} = \mathcal{L},$$

since $[\mathcal{L}_k, \mathcal{L}_k]$ is an ideal of $\mathcal{L}_k$ that must be $\mathcal{L}_k$ or $\{0\}$,
but $\{0\}$ is not possible. #

g) $\mathcal{L}$ = compact $\quad\Rightarrow\quad$ $\mathcal{L}$ = "reductive", i.e. direct sum of an abelian and a semisimple Lie algebra $= \mathcal{L}_{\text{abelian}} \oplus \mathcal{L}_{\text{semisimple}}$.

Chapter 6

Semisimple Lie algebras

6.1 Cartan subalgebra, root vectors, and Cartan–Weyl basis

Consider complex semisimple Lie algebra $\mathcal{L}$ resulting from complexification of a (real!) compact semisimple Lie algebra. (Always assumed in Chapter 6.)

$\hookrightarrow$ W.l.o.g. we can assume:

- structure constants ${f^{AB}}_C$ real,
- generators hermitian: $T^A_{\mathrm{ad}} = \left(T^A_{\mathrm{ad}}\right)^\dagger$,
- Cartan–Killing form g = positiv definite on *real* vector space spanned by $\{T^A\}$.

Construction of "Cartan subalgebra" $\mathcal{H}$

1. Find maximal set $\{H^j\}_{j=1}^r$ of linearly independent T^A_{ad} that mutually commute:

$$[H^j, H^k] = 0, \qquad r \equiv \text{"rank of } \mathcal{L}\text{"} = \text{independent of choice of } \{H^j\}_{j=1}^r, \tag{6.1}$$
$$\mathcal{H} \equiv \text{subalgebra spanned by } \{H^j\},\; r = \dim \mathcal{H}.$$

2. Simultaneous diagonalization of all H^j in adjoint representation:

$$({\mathrm{ad}_{H^j}})^A{}_B = \left(T^j_{\mathrm{ad}}\right)^A{}_B = -\mathrm{i}f^{jA}{}_B \propto \delta^A{}_B \quad \text{for fixed } j. \tag{6.2}$$

$$\Rightarrow\; \mathrm{ad}_{H^j}\left(T^A\right) = \left[H^j, T^A_{\mathrm{ad}}\right] = \mathrm{i}f^{jA}{}_B T^B_{\mathrm{ad}} \propto T^A_{\mathrm{ad}}. \tag{6.3}$$

Renaming $X^a = T^A_{\mathrm{ad}} \notin \mathcal{H}$ in this basis, define

$$\mathrm{ad}_{H^j}\left(X^a\right) = \left[H^j, X^a\right] \equiv \beta^j(a) X^a. \tag{6.4}$$

$\hookrightarrow$ Each generator $X^a \notin \mathcal{H}$ is characterized by a

$$\text{"root vector" } \beta(a) = \big(\beta^1(a), \ldots, \beta^r(a)\big) \neq 0 \quad (0 \text{ would mean } X^a \in \mathcal{H}), \tag{6.5}$$
$$\Phi \equiv \text{set of all root vectors } \beta(a) \neq 0. \tag{6.6}$$

Notation: $E_\beta^{(a)} \equiv X^a$ with $\beta = \beta(a)$.

Comments:

- The generators X^a are *not* hermitian anymore after the diagonalization of all H^j.
- This step requires that the number field of $\mathcal{L}$ is closed.
  $\hookrightarrow$ Take field $\mathbb{C}$, not $\mathbb{R}$!

3. Inspect general $H = h_j H^j \in \mathcal{H}$:

$$[H, X^a] = h_j \left[H^j, X^a\right] = \underbrace{h_j \beta^j(a)}_{\equiv \beta(H) = \text{“linear form” on } \mathcal{H} \text{ (=linear map } \mathcal{H} \to \mathbb{C})} X^a, \tag{6.7}$$

i.e. $\beta \in \mathcal{H}^* =$ dual space of $\mathcal{H}$.

Note: Construction of $\mathcal{H}$ in adjoint representation can be transferred to whole $\mathcal{L}$ if $\mathcal{L}$ is simple, since the adjoint representation is faithful.

Properties of roots:

a) If $\beta(a)$ is a root, then also $-\beta(a)$. $\Rightarrow$ $d_{\mathcal{L}} - r =$ even.

Proof:

$$\begin{aligned} \left[H^j, X^a\right] &= \beta^j(a) X^a, \qquad \left|\ldots^\dagger\right. \quad \text{and use } \beta(a) = \beta(a)^*,\ H^j = \left(H^j\right)^\dagger \\ \left[(X^a)^\dagger, H^j\right] &= \beta^j(a)\,(X^a)^\dagger, \\ \left[H^j, (X^a)^\dagger\right] &= -\beta^j(a)\,(X^a)^\dagger. \end{aligned} \tag{6.8}$$

#

b) If $\beta(a) + \beta(b) \neq 0$, then either $\left[X^a, X^b\right] = 0$,
or $\left[X^a, X^b\right] \neq 0$ is eigenvector to root $\beta(a) + \beta(b)$.

Proof:

$$\begin{aligned} \left[H^j, \left[X^a, X^b\right]\right] &= \left[X^a, \left[H^j, X^b\right]\right] + \left[X^b, \left[X^a, H^j\right]\right] \qquad \text{(Jacobi id.)} \\ &= \beta^j(b)\left[X^a, X^b\right] - \beta^j(a)\left[X^b, X^a\right] \\ &= \underbrace{\left(\beta^j(a) + \beta^j(b)\right)}_{\neq 0 \text{ for some } j\text{-value}} \left[X^a, X^b\right]. \end{aligned} \tag{6.9}$$

$\Rightarrow$ If $\left[X^a, X^b\right] \neq 0$, then it is an eigenvector to root $\beta(a) + \beta(b)$. #

c) $(H^j, X^a) = 0$.

Proof:

$$\begin{aligned} 0 &= \left(\left[H^j, H^k\right], X^a\right) \qquad \left(\text{since } \left[H^j, H^k\right] = 0\right) \\ &= \left(H^j, \left[H^k, X^a\right]\right) = \underbrace{\beta^k(a)}_{\neq 0 \text{ for some } k\text{-value}} \left(H^j, X^a\right). \\ \Rightarrow\ 0 &= \left(H^j, X^a\right). \end{aligned} \tag{6.10}$$

#

d) $\left(X^a, X^b\right) = 0$ if $\beta(a) + \beta(b) \neq 0$.

Proof:

$$\begin{aligned} \left(\left[X^a, H^j\right], X^b\right) &= -\beta^j(a)\left(X^a, X^b\right) \\ = \left(X^a, \left[H^j, X^b\right]\right) &= +\beta^j(b)\left(X^a, X^b\right). \\ \Rightarrow\ 0 &= \underbrace{\left(\beta^j(a) + \beta^j(b)\right)}_{\neq 0 \text{ for some } j\text{-value}} \left(X^a, X^b\right). \\ \Rightarrow\ 0 &= \left(X^a, X^b\right). \end{aligned} \tag{6.11}$$

#

e) $g^{ij} \equiv \mathrm{Tr}\{H^i H^j\}$ in adjoint representation is non-singular and positive definite (=restriction of Cartan–Killing form to $\mathcal{H}$).

$\hookrightarrow$ Define:

$$g^{ij} g_{jk} \equiv \delta^i{}_k, \quad \beta_j \equiv g_{jk}\beta^k, \tag{6.12}$$

$$\underbrace{(\alpha,\beta)}_{\hookrightarrow \text{ positive definite scalar product on the root space } \mathcal{H}^*} \equiv g_{jk}\alpha^j\beta^k = \alpha_k\beta^k. \tag{6.13}$$

Proof:

$$g = \left(g^{AB}\right) = \left(g^{ij}\right) \oplus \left(g^{ab}\right), \quad \text{since } \{H^j\} \perp \{X^a\}.$$

$\Rightarrow$ (g^{ij}) is non-singular and positive definite, since $\left(g^{AB}\right)$ is. #

f) Restricted Killing form calculable from root vectors:

$$(H, H') = \sum_{\alpha\in\Phi} \alpha(H)\alpha(H') \qquad \forall H, H' \in \mathcal{H}. \tag{6.14}$$

Proof: Exercise!

g) All roots $\beta(a)$ are different (no degeneracy of X^a!),
i.e. exactly one eigenvector $E_\beta \equiv E^{(a)}_\beta$ corresponds to a root $\beta(a)$.

Proof in 3 steps:

- Step 1:

$$\left[X^a, X^b\right] \in \mathcal{H} \text{ for } \beta(a) + \beta(b) = 0, \text{ according to proof of b), i.e.}$$

$$\left[X^a, X^b\right] = c_j(a,b)H^j \qquad \big|\, (\ldots, H^k)$$

$$\Rightarrow \left(H^k, \left[X^a, X^b\right]\right) = c_j(a,b)\left(H^j, H^k\right) = c_j(a,b)g^{jk} \equiv c^k(a,b)$$
$$= \left(\left[H^k, X^a\right], X^b\right) = \beta^k(a)\underbrace{\left(X^a, X^b\right)}_{\equiv d(a,b)}. \tag{6.15}$$

Note: $d(a,b) \neq 0$ for at least one pair a, b!
Otherwise $(X^a, X) = 0\ \forall X \in \mathcal{L}$,
i.e. contradiction to non-singularity of metric.

$$\Rightarrow \left[X^a, X^b\right] = \beta_j(a)H^j d(a,b) \neq 0 \quad \text{for some chosen index pair } a, b. \tag{6.16}$$

- Step 2: Choose one specific generator $E^{(a)}_{-\alpha}$ and define subspace $\mathcal{A} \subset \mathcal{L}$:

$$\mathcal{A} \equiv \left[E^{(a)}_{-\alpha}, \mathcal{H}, V_\alpha, \ldots, V_{k\alpha}\right], \tag{6.17}$$

V_α = subspace spanned by all generators $E^{(b)}_\alpha$ with root $\beta(b) = \alpha$,
k = largest integer k, so that $k\alpha$ is a root.

Observation: $\mathcal{A}$ is invariant under multiplication by all generators in $A = \left\{E^{(a)}_{-\alpha}, \mathcal{H}, V_\alpha\right\}$, i.e. $[X, \mathcal{A}] \subseteq \mathcal{A}\ \ \forall X \in A$.

$\hookrightarrow$ Verification by calculating all commutators!

- Step 3: Consider (6.16) on subspace $\mathcal{A}$!

  Identify $\alpha = -\beta(a)$, then $\underbrace{X^a, X^b, \{H^j\}}_{\text{of (6.16)}} \in A$.

  $\Rightarrow$ (6.16) defined also on restriction $\mathcal{A}$ of $\mathcal{L}$.

  $\hookrightarrow$ Evaluate trace of (6.16) on $\mathcal{A}$ in adjoint representation!

  Recall diagonal block structure:

$$\mathrm{ad}_{H^j} = \begin{pmatrix} \overbrace{\beta\cdot\mathbb{1}}^{V_\beta} & & & \\ & \overbrace{0\cdot\mathbb{1}}^{\mathcal{H}} & & \\ & & \overbrace{-\beta\cdot\mathbb{1}}^{V_{-\beta}} & \\ & & & \ddots \end{pmatrix}, \tag{6.18}$$

  because

$$\begin{aligned} \mathrm{ad}_{H^j}\left(H^k\right) &= \left[H^j, H^k\right] = 0, \\ \mathrm{ad}_{H^j}\left(E_\beta^{(b)}\right) &= \left[H^j, E_\beta^{(b)}\right] = \beta^j(b) E_\beta^{(b)}. \end{aligned} \tag{6.19}$$

$$\begin{aligned} \Rightarrow \operatorname*{Tr}_{\mathcal{A}}\{(6.16)\} &= \operatorname*{Tr}_{\mathcal{A}}\left\{\left[X^a, X^b\right]\right\} = 0 \qquad \text{(due to cyclicity!)} \\ &= \operatorname*{Tr}_{\mathcal{A}}\left\{\beta_j(a) H^j d(a,b)\right\} = \beta_j(a) d(a,b)\cdot \operatorname*{Tr}_{\mathcal{A}}\left\{H^j\right\} \\ &= -\alpha_j \cdot d(a,b)\cdot\Big\{\underbrace{-\alpha^j}_{\hookrightarrow E_{-\alpha}^{(a)}} + \underbrace{0}_{\hookrightarrow \mathcal{H}} + \underbrace{\alpha^j\cdot\dim V_\alpha}_{\hookrightarrow V_\alpha} + \ldots + \underbrace{k\alpha^j\cdot\dim V_{k\alpha}}_{\hookrightarrow V_{k\alpha}}\Big\} \end{aligned}$$

$$= -\underbrace{\alpha_j\alpha^j}_{=(\alpha,\alpha)\neq 0}\cdot\underbrace{d(a,b)}_{\neq 0}\cdot\left\{-1 + \sum_{l=1}^{k} l\cdot\underbrace{\dim V_{l\alpha}}_{\geq 0}\right\}. \tag{6.20}$$

  $\Rightarrow$ Unique solution: $k = 1$ with $\dim V_\alpha = 1$. #

$\Rightarrow$ Standard form of a Lie algebra: "Cartan–Weyl basis":

Generators: $H^j, E_\alpha, E_{-\alpha} = (E_\alpha)^\dagger$

with $H^\alpha \equiv \alpha_j H^j$ and $(E_\alpha, E_{-\alpha}) \equiv 1$ (i.e. $d(a,b)$ set to 1).

Commutators:

$$\left[H^j, H^k\right] = 0, \tag{6.21}$$

$$\left[H^j, E_{\pm\alpha}\right] = \pm\alpha^j E_{\pm\alpha}, \tag{6.22}$$

$$[E_\alpha, E_{-\alpha}] = H^\alpha, \tag{6.23}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad \text{if } \alpha+\beta \neq 0, \tag{6.24}$$

$$N_{\alpha\beta} = 0 \text{ if } \alpha+\beta \text{ is not a root.} \tag{6.25}$$

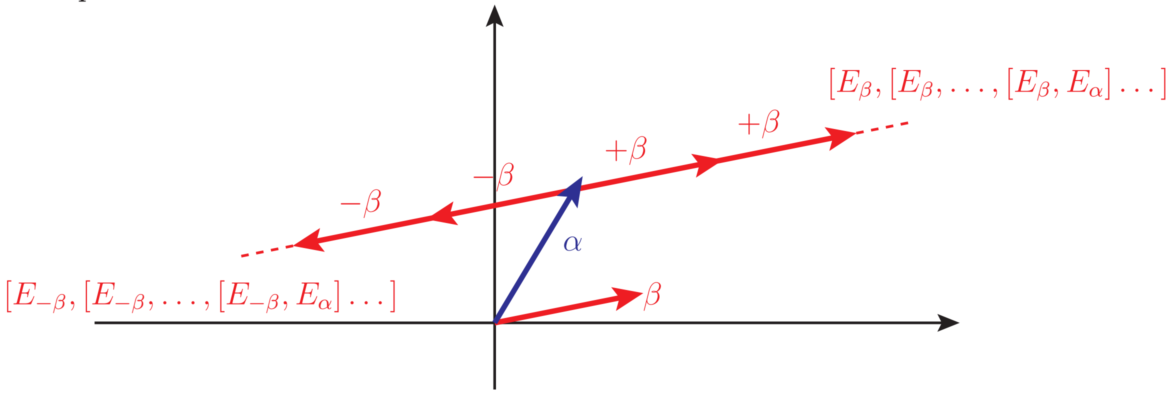
6.2 Geometry of the root system

Root strings:

Definition: "β-string through root α"

$$S_{\beta;\alpha} \equiv \{\text{roots } \alpha + k\beta \mid k = -p, -p+1, \ldots, q;\ p, q \in \mathbb{N}_0, \\ \text{but } \alpha - (p+1)\beta \text{ and } \alpha + (q+1)\beta \text{ are } \textit{not} \text{ roots}\}. \tag{6.26}$$

In root space:



$S_{\beta;\alpha}$ as sl$(2,\mathbb{C})$ representation space:

$S_{\beta;\alpha}$ = representation space of sl$(2,\mathbb{C})$ spanned by $E_{\pm\beta}$, $\beta_j H^j = H^\beta$:

- sl$(2,\mathbb{C})$ algebra:

$$[E_{+\beta}, E_{-\beta}] = H^\beta, \tag{6.27}$$

$$[H^\beta, E_{\pm\beta}] = \pm\beta_j\beta^j E_{\pm\beta} = \pm(\beta,\beta)E_{\pm\beta}. \tag{6.28}$$

- $E_{\pm\beta}$ = shift operators on $S_{\beta;\alpha}$ from $\alpha + k\beta$ to $\alpha + (k \pm 1)\beta$:

$$[E_{\pm\beta}, E_{\alpha+k\beta}] \propto E_{\alpha+(k\pm1)\beta}. \tag{6.29}$$

- $E_{\alpha+k\beta}$ are eigenvectors of $\mathrm{ad}_{\beta_j H^j}$:

$$[H^\beta, E_{\alpha+k\beta}] = \beta_j(\alpha+k\beta)^j E_{\alpha+k\beta} = \underbrace{[(\alpha,\beta) + k(\beta,\beta)]}_{\text{eigenvalues = "weights"}} E_{\alpha+k\beta}. \tag{6.30}$$

$$\Rightarrow \quad \text{highest sl}(2,\mathbb{C}) \text{ weight} = (\alpha,\beta) + q(\beta,\beta) \\ = -(\text{lowest weight}) = -\left[(\alpha,\beta) - p(\beta,\beta)\right]. \tag{6.31}$$

$$\Rightarrow \quad 2\frac{(\alpha,\beta)}{(\beta,\beta)} = p - q \equiv n \in \mathbb{Z}. \tag{6.32}$$

Apply the same arguments to $S_{\alpha;\beta}$ (with p', q' instead of p, q):

$$2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = p' - q' \equiv n' \in \mathbb{Z}. \tag{6.33}$$

⇒ Condition on angle $\theta_{\alpha\beta}$ between roots α, β in root space:

$$0 \leq \cos^2 \theta_{\alpha\beta} \equiv \frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = \frac{nn'}{4} \leq 1. \tag{6.34}$$

In particular, n and n' have the same sign (if both are non-zero)!

Constraints on $S_{\beta;\alpha}$ and $S_{\alpha;\beta}$ from (6.32)–(6.34):

a) Assume special case $\beta = c \cdot \alpha$, $c \in \mathbb{R}$:

$$2\frac{(\alpha,\beta)}{(\beta,\beta)} = \frac{2}{c} = n \in \mathbb{Z} \qquad \Rightarrow \quad |c| = 2, 1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \ldots \tag{6.35}$$

$$2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = 2c = n' \in \mathbb{Z} \qquad \Rightarrow \quad |c| = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \tag{6.36}$$

⇒ 2 possibilities: (w.l.o.g. $|c| \leq 1$)

(i) $|c| = 1$, i.e. $\beta = +\alpha$ or $\beta = -\alpha$.
↪ Nothing new, since $\pm\alpha$ are trivially roots.

(ii) $|c| = \frac{1}{2}$, i.e. $\alpha = +2\beta$ or $\alpha = -2\beta$.
↪ Contradiction to proof of property g) above!

⇒ With α being a root, $\pm\alpha$ are the only multiples of α being roots!

b) Possibilities for $\beta \neq \pm\alpha$ ($0 \leq \cos^2 \theta_{\alpha\beta} < 1$):

n		n'	$\theta_{\alpha\beta}$	length ratio from (6.32)/(6.33): $\sqrt{\frac{(\beta,\beta)}{(\alpha,\alpha)}} = \sqrt{\frac{n'}{n}}$
0	or	0	$\frac{\pi}{2}$	not fixed
+1		+1	$\frac{\pi}{3}$	1
−1		−1	$\frac{2\pi}{3}$	1
+1		+2	$\frac{\pi}{4}$	$\sqrt{2}$
−1		−2	$\frac{3\pi}{4}$	$\sqrt{2}$
+1		+3	$\frac{\pi}{6}$	$\sqrt{3}$
−1		−3	$\frac{5\pi}{6}$	$\sqrt{3}$

+ cases with $\alpha \leftrightarrow \beta$, $n \leftrightarrow n'$

Determination of $|N_{\alpha\beta}|$: $\Big([E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \Big)$

- From definition and $E_{-\alpha} = E_\alpha^\dagger$:

$$N_{\alpha\beta} = -N_{\beta\alpha} = +N^*_{-\beta,-\alpha} = -N^*_{-\alpha,-\beta}. \tag{6.37}$$

- Choose 3 (non-vanishing) roots α, β, γ with $\alpha + \beta + \gamma = 0$:

$$\underbrace{[E_\alpha, [E_\beta, E_\gamma]]}_{\ldots+\text{cyclic}=0} = \big[E_\alpha, N_{\beta\gamma} \underbrace{E_{\beta+\gamma}}_{=E_{-\alpha}}\big] = N_{\beta\gamma}\alpha_j H^j.$$

$$\Rightarrow\ 0 = N_{\beta\gamma}\alpha_j + N_{\gamma\alpha}\beta_j + N_{\alpha\beta} \underbrace{\gamma_j}_{=\,-\alpha_j-\beta_j}, \quad \text{since } \{H^j\} = \text{independent}.$$

$$\Rightarrow\ 0 = \alpha_j(N_{\beta\gamma} - N_{\alpha\beta}) + \beta_j(N_{\gamma\alpha} - N_{\alpha\beta}), \quad \text{but } \alpha, \beta \text{ are independent}.$$

$$\Rightarrow\ N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha}, \quad \text{i.e. } N_{\alpha\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha}. \tag{6.38}$$

- Jacobi identity on root string $S_{\beta;\alpha}$:

$$0 = \big[E_\beta, \underbrace{[E_{-\beta}, E_{\alpha+k\beta}]}_{=\,N_{-\beta,\alpha+k\beta}E_{\alpha+(k-1)\beta}}\big] + \big[E_{-\beta}, \underbrace{[E_{\alpha+k\beta}, E_\beta]}_{=\,N_{\alpha+k\beta,\beta}E_{\alpha+(k+1)\beta}}\big] + \big[E_{\alpha+k\beta}, \underbrace{[E_\beta, E_{-\beta}]}_{=\,\beta_j H^j}\big],$$

$$0 = \big[N_{-\beta,\alpha+k\beta}N_{\beta,\alpha+(k-1)\beta} + N_{\alpha+k\beta,\beta}N_{-\beta,\alpha+(k+1)\beta} - \beta_j(\alpha+k\beta)^j\big] \underbrace{E_{\alpha+k\beta}}_{\neq 0},$$

$$\Rightarrow\ (\alpha,\beta) + k(\beta,\beta) = N_{-\beta,\alpha+k\beta}N_{\beta,\alpha+(k-1)\beta} + N_{\alpha+k\beta,\beta}N_{-\beta,\alpha+(k+1)\beta}.$$

Using

$$N_{-\beta,\alpha+k\beta} \underset{(6.37)}{=} -N^*_{\beta,-\alpha-k\beta} \underset{(6.38)}{=} -N^*_{\alpha+(k-1)\beta,\beta} \underset{(6.37)}{=} N^*_{\beta,\alpha+(k-1)\beta}, \tag{6.39}$$

$$N_{-\beta,\alpha+(k+1)\beta} \underset{(6.38)}{=} N_{-\alpha-k\beta,-\beta} \underset{(6.37)}{=} -N^*_{\alpha+k\beta,\beta}, \tag{6.40}$$

we get the recursive relation

$$(\alpha,\beta) + k(\beta,\beta) = F(k-1) - F(k), \qquad F(k) = |N_{\alpha+k\beta,\beta}|^2. \tag{6.41}$$

- Boundary of recursion (6.41):

$$\begin{aligned} [E_\beta, E_{\alpha+q\beta}] = 0 \quad &\Rightarrow \quad N_{\beta,\alpha+q\beta} = 0 \quad \Rightarrow \quad F(q) = 0, \\ [E_{-\beta}, E_{\alpha-p\beta}] = 0 \quad &\Rightarrow \quad \underbrace{N_{-\beta,\alpha-p\beta}}_{=N_{\beta,\alpha-(p+1)\beta}} = 0 \quad \Rightarrow \quad F(-p-1) = 0. \end{aligned} \tag{6.42}$$

$\Rightarrow$ Unique solution for $F(k)$:

$$\begin{aligned} F(k) &= (q-k)\left[(\alpha,\beta) + \tfrac{1}{2}(k+q+1)(\beta,\beta)\right] \\ &= (q-k)\left[\tfrac{1}{2}(p-q) + \tfrac{1}{2}(k+q+1)\right](\beta,\beta), \\ F(0) &= |N_{\alpha\beta}|^2 = \tfrac{1}{2}q(p+1)(\beta,\beta). \end{aligned} \tag{6.43}$$

Note:
- $N_{\alpha\beta}$ can be chosen real. (If needed, redefine phase of E_α.)
- Sign determination of $N_{\alpha\beta}$ not so trivial, details see below!

Weyl reflections:

Definition:

$$\sigma_\beta(\alpha) \equiv \alpha - 2\frac{(\alpha,\beta)}{(\beta,\beta)}\beta = \text{“Weyl reflection” of } \alpha \text{ w.r.t. the hyperplane } \perp \beta. \tag{6.44}$$

Check properties: (see (6.35))

- $\sigma_\beta(\alpha) = $ root, since $p \leq n = 2\frac{(\alpha,\beta)}{(\beta,\beta)} = p - q$ and $q \geq -n = q - p$.
- Projections:

$$\big(\sigma_\beta(\alpha), \beta\big) = (\alpha,\beta) - n(\beta,\beta) = (\alpha,\beta) - 2\frac{(\alpha,\beta)}{(\beta,\beta)}(\beta,\beta) = -(\alpha,\beta),$$
$$\big(\sigma_\beta(\alpha), \sigma_\beta(\alpha)\big) = (\alpha,\alpha) - 2n(\alpha,\beta) + n^2(\beta,\beta)^2 = (\alpha,\alpha). \tag{6.45}$$

“Weyl group” $\equiv$ group of all Weyl reflections.

$\hookrightarrow$ subgroup of the full symmetry group of the root system (and as such finite).

Note: The finiteness of a reflection group is non-trivial!

Abstract definition of a “root system”:

A “(reduced crystallographic) root system” is a finite set Φ of non-zero vectors (“roots”) in some finite-dimensional real vector space V with scalar product $(. , .)$, with the following properties:

(i) The roots span V.

(ii) For each $\alpha \in \Phi$, $-\alpha$ is the only other multiple of α in Φ.

(iii) Φ is closed under Weyl reflections, i.e. $\sigma_\beta(\alpha) \in \Phi \;\; \forall\, \alpha, \beta \in \Phi$.

(iv) “Integrality”: $2\frac{(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z} \;\; \forall\, \alpha,\beta \in \Phi$.

The “rank” of the root system Φ is defined to be $\dim(V)$.

$\Phi^+ \equiv \{\alpha \in \Phi \mid \alpha > 0\} = $ set of all positive roots.

Φ is “reducible” if it can be decomposed into a sum of mutually orthogonal parts, i.e. if $\Phi = \Phi_1 + \Phi_2$ with $\Phi_i \subset V_i$ and $V = V_1 \oplus V_2$, $V_1 \perp V_2$. Otherwise Φ is “irreducible”.

Note $\Phi = $ reducible $\Leftrightarrow$ $\mathcal{L} = $ semisimple, but *not* simple.

Serre’s theorem:

There is a one-to-one correspondence between abstract root systems and complex semi-simple Lie algebras.

6.3 Simple roots, Cartan matrix, and Chevalley basis

Chevalley relations:

1. Start from auxiliary identity: (Exercise!)

$$\frac{(\alpha+\beta,\alpha+\beta)}{(\alpha,\alpha)} = \frac{p+1}{q} \quad \text{for roots } \alpha,\ \beta \text{ if } \alpha+\beta = \text{root}. \tag{6.46}$$

   Outline of proof: (Exercise!)
   Use $p = 2\frac{(\alpha,\beta)}{(\beta,\beta)} + q$ in auxiliary quantity

$$\begin{aligned} M \equiv p - \frac{(\alpha+\beta,\alpha+\beta)}{(\alpha,\alpha)}q + 1 &= \left(1 - \frac{(\beta,\beta)}{(\alpha,\alpha)}q\right)\left(1 + 2\frac{(\alpha,\beta)}{(\beta,\beta)}\right) \\ &= \left(1 - \frac{n'}{n}q\right)(1+n) \end{aligned} \tag{6.47}$$

   and show that $M = 0$ for all possible cases of n, n' ... #

2. Application of (6.46) to $N_{\alpha\beta}$ for $\alpha+\beta = $ root:

$$\begin{aligned} |N_{\alpha\beta}|^2 &= \frac{1}{2}q(p+1)(\beta,\beta)\cdot\frac{p+1}{q}\cdot\frac{(\alpha,\alpha)}{(\alpha+\beta,\alpha+\beta)} \\ &= \frac{1}{2}(p+1)^2\frac{(\alpha,\alpha)(\beta,\beta)}{(\alpha+\beta,\alpha+\beta)}. \end{aligned} \tag{6.48}$$

3. Redefinition of generators:

$$e_\alpha \equiv \sqrt{\frac{2}{(\alpha,\alpha)}}\,E_\alpha, \qquad h_\alpha \equiv \frac{2}{(\alpha,\alpha)}\,\alpha_j H^j. \tag{6.49}$$

⇒ "Chevalley relations":

$$\begin{aligned} &[h_\alpha, h_\beta] = 0, && [h_\beta, e_{\pm\alpha}] = \pm 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\,e_{\pm\alpha}, \\ &[e_\alpha, e_{-\alpha}] = h_\alpha, && [e_\alpha, e_\beta] = \begin{cases} \pm(p+1)e_{\alpha+\beta} & \text{if } \alpha+\beta = \text{root}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{6.50}$$

Comments:

- In this basis, all structure constants are integers.
- The sign choice in the last relation is non-trivial.
  ↪ Details clarified below!

"Positive" and "simple" roots:

- A root α is "positive" ("negative") if the first non-vanishing component α^j of the root vector in the fixed order of $H^1, \ldots, H^r$ is positive (negative).
- A root α is "simple" if α is positive and cannot be written as linear combination of other roots with positive coefficients.

Properties of simple roots $\boldsymbol{\alpha^{(i)}}$:

- Differences of simple roots cannot be roots. $\quad (p^{(i)} = p^{(j)} = 0)$

$$\Rightarrow \; \left(\alpha^{(i)}, \alpha^{(j)}\right) \leq 0, \quad \text{i.e. } \angle\left(\alpha^{(i)}, \alpha^{(j)}\right) = \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}. \tag{6.51}$$

- There are only 4 possible non-trivial chains for two simple roots:

$$\alpha^{(i)}, \alpha^{(i)} + \alpha^{(j)}, \ldots, \alpha^{(i)} + q\alpha^{(j)}, \quad q = 0, 1, 2, 3.$$

- There are exactly $r = \text{rank}(\mathcal{L})$ simple roots, and they span the whole root space.
- Any (positive) root β is a linear combination of simple roots with integer (positive) coefficients:

$$\beta = b_i \alpha^{(i)}, \qquad \sum_{i=1}^{r} b_i \equiv \text{ht}(\beta) = \textit{height} \text{ of root } \beta. \tag{6.52}$$

Two new bases: simple coroots and fundamental weights

↪ Particularly relevant in representation theory!

- To each root α define a coroot $\breve{\alpha}$:

$$\breve{\alpha} \equiv \frac{2\alpha}{(\alpha, \alpha)}. \tag{6.53}$$

  "Simple coroots":

$$\breve{\alpha}^{(i)} \equiv \frac{2\alpha^{(i)}}{\left(\alpha^{(i)}, \alpha^{(i)}\right)}, \qquad i = 1, \ldots, r. \tag{6.54}$$

  $\Rightarrow \; \mathcal{B} \equiv \{\breve{\alpha}^{(i)}\}_{i=1}^{r}$ is a basis of $\mathcal{H}^*$.

- "Dynkin basis" of $\mathcal{H} \equiv$ dual basis to $\mathcal{B} \equiv \mathcal{B}^* = \{\Lambda_{(i)}\}_{i=1}^{r}$.

$$\left(\breve{\alpha}^{(i)}, \Lambda_{(j)}\right) = \delta^i_j, \qquad \Lambda_{(j)} = \text{"fundamental weights"}. \tag{6.55}$$

- Some relations:

$$\alpha = a_i \alpha^{(i)} = \breve{a}_i \breve{\alpha}^{(i)}, \quad \breve{a}_i = \left(\alpha, \Lambda_{(i)}\right) = \frac{a_i}{2}\left(\alpha^{(i)}, \alpha^{(i)}\right), \tag{6.56}$$

$$\lambda = \lambda^j \Lambda_{(j)}, \quad \lambda^j = \left(\lambda, \breve{\alpha}^{(j)}\right) = 2\frac{\left(\lambda, \alpha^{(j)}\right)}{\left(\alpha^{(j)}, \alpha^{(j)}\right)} = \text{"Dynkin labels" of } \lambda, \tag{6.57}$$

$$(\alpha, \lambda) = \breve{a}_i \lambda^i = \sum_{i=1}^{r} \frac{1}{2} a_i \lambda^i \left(\alpha^{(i)}, \alpha^{(i)}\right). \tag{6.58}$$

Cartan matrix and Chevalley basis:

- "Cartan matrix" A of $\mathcal{L}$

$$A^{ij} \equiv 2\frac{\left(\alpha^{(i)},\alpha^{(j)}\right)}{\left(\alpha^{(j)},\alpha^{(j)}\right)} = \left(\alpha^{(i)},\check{\alpha}^{(j)}\right). \tag{6.59}$$

$$\Rightarrow\ A = \begin{pmatrix} 2 & A^{12} & \cdots & & \\ A^{21} & 2 & A^{23} & \cdots & \\ \vdots & \vdots & 2 & & \\ & & & \ddots & \\ & & & & 2 \end{pmatrix} \qquad \text{with } A^{ij} = \text{integer} \leq 0 \text{ for } i \neq j. \tag{6.60}$$

Note: ith row of A = components of $\alpha^{(i)}$ in Dynkin basis.

- "Chevalley basis" $\equiv \{h_{\alpha^{(i)}}\} \cup \{e_{\alpha^{(i)}}\}$.

$$[h_{\alpha^{(i)}}, e_{\pm\alpha^{(j)}}] = \pm A^{ji} e_{\pm\alpha^{(j)}}, \tag{6.61}$$

$$[e_{\alpha^{(i)}}, e_{\alpha^{(j)}}] = \pm e_{\alpha^{(i)}+\alpha^{(j)}} \text{ or } 0, \quad \text{if } \alpha^{(i)}+\alpha^{(j)} \text{ is root or not.} \tag{6.62}$$

$\hookrightarrow$ Signs fixed by convention, e.g. "+" for $\alpha^{(i)} < \alpha^{(j)}$.

Serre relations:

$$\left(\mathrm{ad}\, e_{\pm\alpha^{(i)}}\right)^{1-A^{ji}} e_{\pm\alpha^{(j)}} = 0. \tag{6.63}$$

Proof:

$$1 - A^{ji} = 1 - 2\frac{\left(\alpha^{(j)},\alpha^{(i)}\right)}{\left(\alpha^{(i)},\alpha^{(i)}\right)} = 1 - n_{ij} = 1 + q_{ij} = \text{smallest positive integer } k \text{ so that } \alpha^{(j)} + k\alpha^{(i)} \text{ is not a root.}$$

#

Simple properties of Cartan matrices:

$$\begin{aligned} &\text{a)} \quad A^{ii} = 2, && (6.64)\\ &\text{b)} \quad A^{ij} = 0 \;\Leftrightarrow\; A^{ji} = 0, && (6.65)\\ &\text{c)} \quad A^{ij} \in \{0, -1, -2, -3\} \text{ for } i \neq j, && (6.66)\\ &\text{d)} \quad \text{if } A^{ij} \in \{-2, -3\} \text{ then } A^{ji} = -1 \text{ for } i \neq j, && (6.67)\\ &\text{e)} \quad \det(A) > 0. && (6.68) \end{aligned}$$

Proof:

- a) and b) obvious from definition of A.
- c) and d) follow from properties of root strings (see Section 6.2):
  $n, n' < 0$, since $p = p' = 0$ because of simplicity of roots $\alpha^{(i)}$, $\alpha^{(j)}$.
- To prove e), factorize A into diagonal matrix D and "Gram matrix" S:

$$D = \operatorname{diag}(d_1, \ldots, d_r), \quad d_j = 2/\big(\alpha^{(j)}, \alpha^{(j)}\big) > 0, \quad \det(D) > 0,$$

$$S = (s_{ij}), \quad s_{ij} = \left(\alpha^{(i)}, \alpha^{(j)}\right), \quad \det(S) > 0, \text{ since } \{\alpha^{(i)}\} \text{ are linearly independent.}$$

$$\Rightarrow \quad \det(A) = \det(SD) = \det(S) \cdot \det(D) > 0. \qquad \#$$

Examples:

$$A_{\mathrm{sl}(2)} = (\,2\,), \qquad A_{\mathrm{sl}(3)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad A_{\mathrm{sl}(4)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{6.69}$$

Relation between A and (semi)simplicity of $\mathcal{L}$:

- Isomorphic semisimple Lie algebras have the same matrix A up to some possible renumbering of simple roots (rows/columns).
- $\mathcal{L}$ is *not* simple: $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, with $\mathcal{L}_i$ = semisimple Lie subalgebras of $\mathcal{L}$.
  $\hookrightarrow \;[X_1, X_2] = 0 \;\; \forall\, X_i \in \mathcal{L}_i$.
  $\Leftrightarrow$ A is "reducible" to the following block form by renumbering of roots

$$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right), \qquad A_i = \text{Cartan matrix of } \mathcal{L}_i. \tag{6.70}$$

Reconstruction of all simple roots from A:

- Ratios of root lengths l_i: $\frac{l_i}{l_j} = \sqrt{\frac{A^{ij}}{A^{ji}}}$.
- Angles θ_{ij} between roots: $\cos\theta_{ij} = -\frac{1}{2}\sqrt{A^{ij}A^{ji}}$.

$\Rightarrow$ Simple roots determined up to orientation and overall normalization (=convention).

Examples:

a) sl(3): $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Known: $l_1 = l_2$, $\cos\theta_{12} = -\frac{1}{2}$, i.e. $\theta_{12} = \frac{2\pi}{3}$.

Definable: $l_1 \equiv 1$, $\alpha^{(1)} \equiv \vec{e}_1$, $\alpha^{(2)} \cdot \vec{e}_2 > 0$.

$\hookrightarrow$ $\alpha^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha^{(2)} = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$.

b) G_2: $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

Known: $l_1 = \sqrt{3}l_2$, $\cos\theta_{12} = -\frac{1}{2}\sqrt{3}$, i.e. $\theta_{12} = \frac{5\pi}{6}$.

Definable: $l_2 \equiv 1$, $\alpha^{(1)} \equiv \sqrt{3}\,\vec{e}_2$, $\alpha^{(2)} \cdot \vec{e}_1 > 0$.

$\hookrightarrow$ $\alpha^{(1)} = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$, $\alpha^{(2)} = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$.

Reconstruction of the full root system from A: ("Serre construction")

Idea:

Each root $\alpha > 0$ is a unique combination $\alpha = a_i \alpha^{(i)}$ with a_i = integer ≥ 0 and corresponds exactly to one shift operator e_α, which is an eigenvector of all $\mathrm{ad}_{h_{\alpha^{(j)}}}$:

$$\mathrm{ad}_{h_{\alpha^{(j)}}} e_\alpha = \frac{2(\alpha^{(j)}, \alpha)}{(\alpha^{(j)}, \alpha^{(j)})} e_\alpha = a_i A^{ij} e_\alpha .$$

$\hookrightarrow$ Each $\alpha > 0$ can be obtained upon recursively constructing all possible root strings of all simple roots $\alpha^{(i)}$, starting from the simple roots themselves, and characterized by the Dynkin labels $a_i A^{ij}$.

Recursive algorithm:

1. Roots of height 1:
   These are the simple roots $\alpha^{(i)}$, which are known to exist.
   Recall (6.61): $\mathrm{ad}_{h_{\alpha^{(j)}}} e_{\alpha^{(i)}} = A^{ij} e_{\alpha^{(i)}}$.
   $\hookrightarrow$ Simple root $e_{\alpha^{(i)}}$ is eigenvector to $h_{\alpha^{(j)}}$ with eigenvalues A^{ij}.
   $\hookrightarrow$ $e_{\alpha^{(i)}}$ is represented by its "weight vector" $|A^{i1}, \ldots, A^{ir}\rangle$ of Dynkin labels.
2. Roots of height 2:
   Consider all root strings of $e_{\alpha^{(k)}}$ through $e_{\alpha^{(i)}}$:
   - $\alpha^{(i)} - \alpha^{(k)}$ is never a root, i.e. $\mathrm{ad}_{e_{-\alpha^{(k)}}} e_{\alpha^{(i)}} = 0$,
   - Serre relations: $\left(\mathrm{ad}\, e_{\alpha^{(k)}}\right)^{1-A^{ik}} e_{\alpha^{(i)}} = 0$.

   $\hookrightarrow$ Root strings start at $\alpha^{(i)}$ and have lengths $1 - A^{ik}$ in $\alpha^{(k)}$ direction, and $\alpha^{(i)} + \alpha^{(k)}$ is a root (i.e. $e_{\alpha^{(i)}+\alpha^{(k)}} \neq 0$) exactly if $-A^{ik} > 0$.
   $\Rightarrow$ All roots of height 2 through $\alpha^{(i)}$ determined and represented by $|A^{i1} + A^{k1}, \ldots, A^{ir} + A^{kr}\rangle$.
3. Roots of height $(n+1)$ from roots of height n (starting with $n = 2$):
   Consider all root strings of $e_{\alpha^{(k)}}$ through root $\beta = b_i \alpha^{(i)}$ with $\mathrm{ht}(\beta) = n$:
   $\beta - p\alpha^{(k)}, \ldots, \beta, \ldots, \beta + q\alpha^{(k)}$.
   - p can be read from roots of lower weight.
   - Recall (6.32): $p - q = 2\dfrac{(\beta, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})} = b_i A^{ik}, \quad q = p - b_i A^{ik}$.

   $\hookrightarrow$ $\beta + \alpha^{(k)}$ is root if $q > 0$.
   $\Rightarrow$ All roots of height $(n+1)$ through β determined and represented by $|A^{k1} + b_i A^{i1}, \ldots, A^{kr} + b_i A^{ir}\rangle$.
   Repeat this step until no roots with bigger height are possible.
4. Adding for each positive root α the negative root $-\alpha$ completes the set Φ of roots.

Extension to reconstruct the whole algebra:

Chevalley relations (6.50) fix algebra up to signs in

$$[e_\alpha, e_\beta] = \pm(p+1)e_{\alpha+\beta} \quad \text{if } \alpha+\beta = \text{root}.$$

Sign choice [3]: Free sign choice for all "extra special pairs" of roots α, β, the others follow from algebra.

- An ordered pair $\{\alpha, \beta\}$ is "special" if $\alpha + \beta = \text{root}$ and $\alpha < \beta$;
- a special pair $\{\alpha, \beta\}$ is "extra special" if $\alpha < \alpha'$ for all special pairs $\{\alpha', \beta'\}$ with $\alpha' + \beta' = \alpha + \beta$.

Examples:

a) sl(3): $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

- Height 1: 2 simple roots: $\alpha^{(1)} \to |2, -1\rangle$, $\alpha^{(2)} \to |-1, 2\rangle$.
- Height 2: 2 relevant Serre relations for $i \neq j$:

$$\left(\text{ad}_{e_{\alpha^{(1)}}}\right)^{1-A^{21}} e_{\alpha^{(2)}} = \left(\text{ad}_{e_{\alpha^{(1)}}}\right)^{2} e_{\alpha^{(2)}} = [e_{\alpha^{(1)}}, [e_{\alpha^{(1)}}, e_{\alpha^{(2)}}]] = 0$$

$$\Rightarrow \alpha_3 \equiv \alpha^{(1)} + \alpha^{(2)} = \text{root}, \quad e_{\alpha_3} \equiv +\left[e_{\alpha^{(1)}}, e_{\alpha^{(2)}}\right].$$

$$\left(\text{ad}_{e_{\alpha^{(2)}}}\right)^{1-A^{12}} e_{\alpha^{(1)}} = \cdots = 0$$

$\Rightarrow$ no new information.

$\Rightarrow \alpha_3 \to |A^{11} + A^{21}, A^{12} + A^{22}\rangle = |1, 1\rangle$ is the only root of height 2.

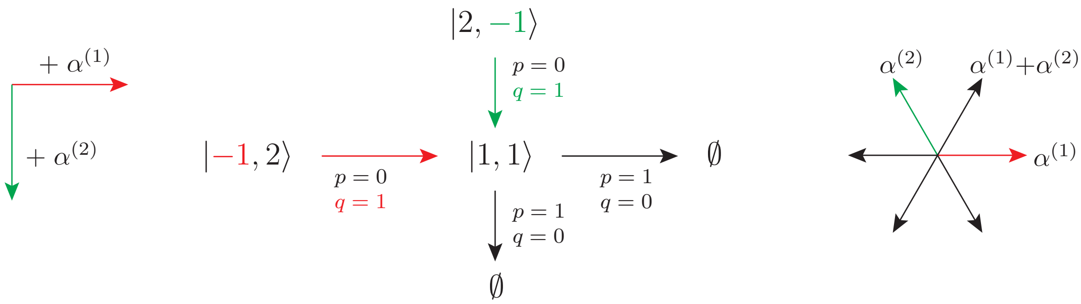
- Height ≥ 3: check 2 strings through α_3:

$$\alpha^{(1)} \text{ string:} \quad p = 1, \quad q = p - (A^{11} + A^{12}) = 0,$$
$$\alpha^{(2)} \text{ string:} \quad p = 1, \quad q = p - (A^{21} + A^{22}) = 0.$$

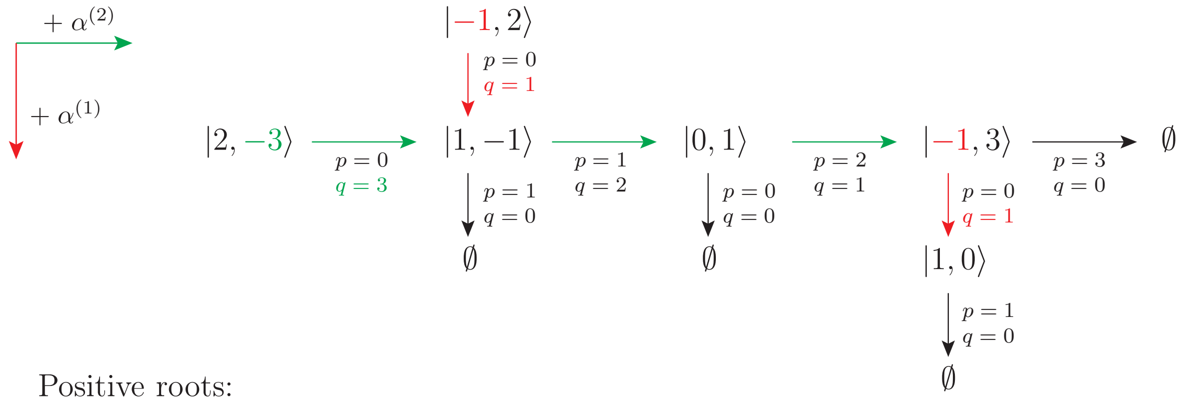
$\Rightarrow$ No roots of height 3!

$\Phi = \{\alpha^{(1)}, \alpha^{(2)}, \alpha_3, -\alpha^{(1)}, -\alpha^{(2)}, -\alpha_3\}$.

Graphical illustration: (coordinates of $\alpha^{(k)}$ see above)



b) G_2: $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

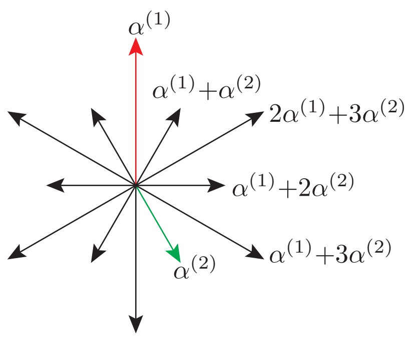


Positive roots:

$\lvert 2,-3\rangle$:	$\alpha^{(1)}$,	$e_{\alpha^{(1)}}$,
$\lvert -1,2\rangle$:	$\alpha^{(2)}$,	$e_{\alpha^{(2)}}$,
$\lvert 1,-1\rangle$:	$\alpha_3 \equiv \alpha^{(1)} + \alpha^{(2)}$,	$e_{\alpha_3} \equiv +\mathrm{ad}_{e_{\alpha^{(1)}}}\, e_{\alpha^{(2)}} = -\mathrm{ad}_{e_{\alpha^{(2)}}}\, e_{\alpha^{(1)}}$,
$\lvert 0,1\rangle$:	$\alpha_4 \equiv \alpha^{(1)} + 2\alpha^{(2)}$,	$e_{\alpha_4} \equiv +\mathrm{ad}_{e_{\alpha^{(2)}}}\, e_{\alpha_3}$,
$\lvert -1,3\rangle$:	$\alpha_5 \equiv \alpha^{(1)} + 3\alpha^{(2)}$,	$e_{\alpha_5} \equiv +\mathrm{ad}_{e_{\alpha^{(2)}}}\, e_{\alpha_4}$,
$\lvert 1,0\rangle$:	$\alpha_6 \equiv 2\alpha^{(1)} + 3\alpha^{(2)}$,	$e_{\alpha_6} \equiv +\mathrm{ad}_{e_{\alpha^{(1)}}}\, e_{\alpha_5}$.

Note: α_3 is the only non-simple root corresponding to a special and an extra special pair of roots.
A root with special *and* extra special pairs correspond to alternative paths for their construction.

The full root system: (coordinates of $\alpha^{(k)}$ see above)



6.4 Classification of complex (semi)simple Lie algebras – Dynkin diagrams

Semisimple complex Lie algebras, root systems, and Cartan matrices:

There is one-to-one correspondences between:

- semisimple complex Lie algebras $\mathcal{L}$,
- abstract root systems Φ with Cartan matrices A.

Similarly, there is one-to-one correspondences between:

- *simple* complex Lie algebras $\mathcal{L}$,
- *irreducible* root systems Φ, with *irreducible* Cartan matrices A.

Decomposition of semisimple complex $\mathcal{L}$:

$$\mathcal{L} = \oplus_i \mathcal{L}_i \qquad \mathcal{L}_i = \text{simple}. \tag{6.71}$$

Simple components $\mathcal{L}_i$ correspond to Φ_i and A_i:

$$\Phi = \cup_i \Phi_i \qquad \Phi_i = \text{irreducible}, \qquad \Phi_i \cap \Phi_j = \emptyset \ \ \forall i \neq j, \tag{6.72}$$

$$A = \oplus_i A_i, \qquad A_i = \text{irreducible}. \tag{6.73}$$

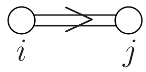
$\Rightarrow$ Classification of simple complex Lie algebras:

- automatically provides a classification of semisimple complex Lie algebras,
- corresponds to a classification of irreducible root systems, which have irreducible Cartan matrices.

"Dynkin diagrams"

$\hookrightarrow$ graphically illustrate Cartan matrices (and thus the corresponding Φ and $\mathcal{L}$).

Graphical rules: $r = \dim(A) = \#(\text{simple roots})$.

- Draw a circle $\circ$ for each simple root (labelled by $i = 1, \dots, r$).
- Connect the two circles i and j by $\max\{|A^{ij}|, |A^{ji}|\}$ lines.
- If $\big(\alpha^{(i)}, \alpha^{(i)}\big) > \big(\alpha^{(j)}, \alpha^{(j)}\big)$ for the two connected roots i and j, then put the ordering sign $>$ on the line(s) between i and j, e.g.: 

  Note: Singly-connected roots have identical lengths;
  different lengths occur for 2 or 3 connecting lines.

$\Rightarrow$ *Connected* Dynkin diagrams correspond to *simple* complex Lie algebras.

Examples:

sl(2) sl(3) sl(4)

Classification simple complex Lie algebras (connected Dynkin diagrams):

Preparation:

- Deconstruction of root systems / Lie algebras:
  Removing a simple root (e.g. number i) from the root system (eliminating row i and column i from A), leads to an allowed simple or semisimple Lie algebra of rank $r-1$.

- Use normalized roots $\hat{\alpha}^{(i)} \equiv \dfrac{\alpha^{(i)}}{\sqrt{(\alpha^{(i)}, \alpha^{(i)})}}$, so that $\left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(i)}\right) = 1$ and

$$\begin{aligned} l_{ij} &\equiv 2\left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(j)}\right) \leq 0, \\ l_{ij}^2 &= \ \#(\text{lines connecting } i \text{ and } j) \ \in \{0,1,2,3\} \quad \text{for} \ \ i \neq j. \end{aligned} \tag{6.74}$$

Restrictions on diagrams:

a) In a set K of k roots, the number L_K of connected pairs of roots is at most $k-1$.
   Proof: Define $\alpha = \sum_{i \in K} \hat{\alpha}^{(i)}$, so that

$$0 < (\alpha, \alpha) = \sum_{i \in K} \left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(i)}\right) + \sum_{\substack{i<j \\ i,j \in K}} 2\left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(j)}\right) = k + \sum_{\substack{i<j \\ i,j \in K}} l_{ij}.$$

$$\Rightarrow \ k \ > \ \sum_{\substack{i<j \\ i,j \in K}} (-l_{ij}) \ \geq \ L_K. \quad \Rightarrow \ L_K \ \leq \ k-1.$$

#

b) There are no Dynkin diagrams with closed cycles (loops).
   Proof: This follows directly from a). #

c) No more than 3 lines can originate from a single root.
   Proof: Let $\hat{\alpha}^{(i)}$ be a normalized root connected to the k roots $\hat{\alpha}^{(j)}$ of the subset K:

$$\begin{aligned} 1 &= \left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(i)}\right) = \left(\hat{\alpha}^{(j)}, \hat{\alpha}^{(j)}\right), \quad \left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(j)}\right) < 0, \quad j \in K, \\ 0 &= \left(\hat{\alpha}^{(j)}, \hat{\alpha}^{(l)}\right), \quad j, l \in K, \end{aligned}$$

   where the last condition stems from the absense of loops.
   The linear independence of the simple roots implies that

$$\begin{aligned} 0 \neq \beta &\equiv \hat{\alpha}^{(i)} - \sum_{j \in K} \left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(j)}\right) \hat{\alpha}^{(j)}, \\ 0 < (\beta, \beta) &= 1 - \sum_{j \in K} \left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(j)}\right)^2 = 1 - \sum_{j \in K} l_{ij}^2/4. \end{aligned}$$

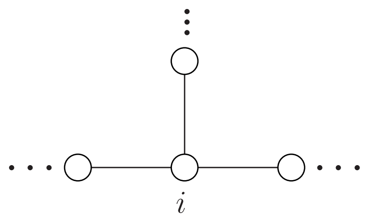
$$\Rightarrow \ 4 > \sum_{j \in K} l_{ij}^2 = \#(\text{lines connected to } i).$$

#

Implications of property c) for a 3-fold-connected root i:

- Only 1 diagram possible with a triple line: 
- 2 possible substructures for a root i with a double and a single line:

- 1 substructure for a root i with 3 single lines:



$\hookrightarrow$ Limitations on lengths of chains indicated by " $\cdots$ " (= one or no line)?

d) "Shrinking rule": Replacing a linear chain of singly-connected roots by one root generates a valid Dynkin diagram.

Sketch of proof: Label the k singly-connected roots $\hat{\alpha}^{(i)}$ by $i = 1, \ldots, k$, so that

$$\left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(i+1)}\right) = -\frac{1}{2}, \quad i = 1, \ldots, k-1,$$
$$\left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(j)}\right) = 0, \quad i, j = 1, \ldots, k-1, \quad |i-j| > 1.$$

Define $\hat{\alpha} = \sum_{i=1}^{k-1} \hat{\alpha}^{(i)}$, which is a unit vector,

$$(\hat{\alpha}, \hat{\alpha}) = \sum_{i=1}^{k} \left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(i)}\right) + 2\sum_{i=1}^{k-1} \left(\hat{\alpha}^{(i)}, \hat{\alpha}^{(i+1)}\right) = k - (k-1) = 1, \tag{6.75}$$

and replace the whole chain $C = \{\hat{\alpha}^{(i)}\}_{i=1}^{k}$ by $\hat{\alpha}$ to get a new Dynkin diagram.

To show: The set $\{\hat{\alpha}\} \cup \{\hat{\alpha}^{(i)}\}_{i=k+1}^{r}$ generates a root system Φ' of rank $r - k + 1$.

- Linear independence of $\{\hat{\alpha}\} \cup \{\hat{\alpha}^{(i)}\}_{i=k+1}^{r}$ and rank of Φ' obviously ok.
- Check angles between simple roots:
  Note that any root $\hat{\beta} \in \{\hat{\alpha}^{(i)}\}_{i=k+1}^{r}$ not in C could be connected to only one root $\hat{\alpha}^{(j)} \in C$, since there is no loop. But $\hat{\beta}$ has the same non-trivial angle (i.e. $\neq \pi/2$) with $\hat{\alpha}^{(j)}$ and the new root $\hat{\alpha}$:

$$(\hat{\beta}, \hat{\alpha}) = \sum_{i=1}^{k-1} \left(\hat{\beta}, \hat{\alpha}^{(i)}\right) = \left(\hat{\beta}, \hat{\alpha}^{(j)}\right).$$

  $\hookrightarrow$ $\hat{\alpha}^{(j)}$ can be replaced by $\hat{\alpha}$ in all scalar products with $\hat{\beta}$.
  $\Rightarrow$ Integrality and Weyl reflections ok!
- Show non-existence of multiples of roots other than $\pm\alpha$ yourself?

e) A Dynkin diagram contains at most one double line.

Proof: According to c), two roots with double lines could only be linked by a chain of singly-connected roots. Shrinking this chain to a single root as in d), would lead to a root with 4 lines attached. → Contradiction! #

f) There are only 3 possible structures with a double line:

Proof: Consider 2 singly-connected chains $\{\hat{\alpha}^{(j)}\}_{j=1}^{n}$ and $\{\hat{\beta}^{(k)}\}_{k=1}^{m}$ with a double line linking $\hat{\alpha}^{(n)}$ and $\hat{\beta}^{(m)}$, where $\hat{\beta}^{(k)}$ are just some renamed roots $\hat{\alpha}^{(i)}$, so that

$$\left(\hat{\alpha}^{(j)}, \hat{\alpha}^{(j+1)}\right) = \left(\hat{\beta}^{(k)}, \hat{\beta}^{(k+1)}\right) = -\frac{1}{2}, \quad j = 1, \ldots, n-1, \;\; k = 1, \ldots, m-1,$$
$$\left(\hat{\alpha}^{(n)}, \hat{\beta}^{(m)}\right) = -\frac{1}{\sqrt{2}}, \quad \left(\hat{\alpha}^{(j)}, \hat{\beta}^{(k)}\right) = 0, \quad j \neq k, \;\; j = 1, \ldots, n, \;\; k = 1, \ldots, m.$$

Analyze the scalar products of the vectors $\alpha \equiv \sum_{j=1}^{n} j\hat{\alpha}^{(j)}$ and $\beta \equiv \sum_{k=1}^{m} k\hat{\beta}^{(k)}$,

$$(\alpha, \alpha) = \sum_{j=1}^{n} j^2 - \sum_{j=1}^{n-1} j(j+1) = \frac{n(n+1)}{2},$$
$$(\beta, \beta) = \sum_{k=1}^{m} k^2 - \sum_{k=1}^{m-1} k(k+1) = \frac{m(m+1)}{2},$$
$$(\alpha, \beta) = (\alpha^{(n)}, \beta^{(m)}) = -\frac{mn}{\sqrt{2}}.$$

Schwartz's inequality implies a condition on n and m:

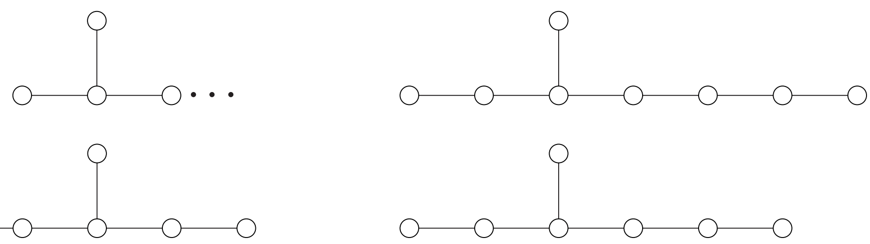
$$0 < (\alpha, \alpha)(\beta, \beta) - (\alpha, \beta)^2 = \frac{mn(m+1)(n+1)}{4} - \frac{m^2n^2}{2} = \frac{mn(1+m+n-mn)}{4}.$$
$$\Rightarrow \; (m-1)(n-1) < 2.$$

Note that equality is ruled out, because α and β are linearly independent.

The 3 different types of solutions for $n, m \geq 1$ correspond to the above diagrams, assuming that the $\alpha^{(i)}$ are longer than $\beta^{(j)}$ (unnormalized roots):

- $m = n = 2$: diagram on the right.
- $m = 1, n \in \mathbb{N}$: diagram on the left.
- $n = 1, m \in \mathbb{N}$: diagram in the middle.

g) There are only 4 different types of diagrams with a root connected to 3 other roots:



Proof: Consider 3 singly-connected chains $\{\hat{\alpha}^{(j)}\}_{j=1}^{n-1}$, $\{\hat{\beta}^{(k)}\}_{k=1}^{m-1}$, and $\{\hat{\gamma}^{(l)}\}_{l=1}^{p-1}$ which are linked to the root $\hat{\delta}$ by $\hat{\alpha}^{(n-1)}$, $\hat{\beta}^{(m-1)}$, and $\hat{\gamma}^{(p-1)}$.

As in f), analyze the scalar products of the vectors $\alpha \equiv \sum_{j=1}^{n-1} j\hat{\alpha}^{(j)}$, $\beta \equiv \sum_{k=1}^{m-1} k\hat{\beta}^{(k)}$, and $\gamma \equiv \sum_{l=1}^{p-1} l\hat{\gamma}^{(l)}$:

$$
\begin{aligned}
(\alpha,\alpha) &= \frac{n(n-1)}{2}, & (\hat{\delta},\alpha) &= (n-1)\,(\hat{\delta},\alpha^{(n-1)}) = -\frac{n-1}{2},\\
(\beta,\beta) &= \frac{m(m-1)}{2}, & (\hat{\delta},\beta) &= (m-1)\,(\hat{\delta},\beta^{(m-1)}) = -\frac{m-1}{2},\\
(\gamma,\gamma) &= \frac{p(p-1)}{2}, & (\hat{\delta},\gamma) &= (p-1)\,(\hat{\delta},\gamma^{(p-1)}) = -\frac{p-1}{2}.
\end{aligned}
$$

Calculate the norm of the vector

$$
\epsilon \equiv \hat{\delta} - \frac{(\hat{\delta},\alpha)}{(\alpha,\alpha)}\alpha - \frac{(\hat{\delta},\beta)}{(\beta,\beta)}\beta - \frac{(\hat{\delta},\gamma)}{(\gamma,\gamma)}\gamma \neq 0,
$$

which is orthogonal to α, β, γ,

$$
\begin{aligned}
&0 < (\epsilon,\epsilon) = 1 - \frac{(\hat{\delta},\alpha)^2}{(\alpha,\alpha)} - \frac{(\hat{\delta},\beta)^2}{(\beta,\beta)} - \frac{(\hat{\delta},\gamma)^2}{(\gamma,\gamma)} = \frac{1}{2}\left(\frac{1}{m} + \frac{1}{n} + \frac{1}{p} - 1\right).\\
\Rightarrow\ &1 < \frac{1}{m} + \frac{1}{n} + \frac{1}{p}.
\end{aligned}
$$

The 4 different types of solutions for $n, m, p > 1$ correspond to the above diagrams:

- $m = n = 2,\ \ 1 < p \in \mathbb{N}$: upper left diagram.
- $m = 2, n = 3, p = 5$: upper right diagram.
- $m = 2, n = 3, p = 4$: lower right diagram.
- $m = 2, n = 3, p = 3$: lower left diagram.

h) Finally, there is no restriction on diagrams with only one singly-connected chain without bifurcations.

Survey of all finite-dimensional simple complex Lie algebras

$\hookrightarrow$ 4 infinite series of "classical Lie algebras" (r = rank)

- $A_r \equiv \mathrm{sl}(r+1,\mathbb{C}),\ \ r \geq 1$,
- $B_r \equiv \mathrm{so}(2r+1,\mathbb{C}),\ \ r \geq 3$,
- $C_r \equiv \mathrm{sp}(2r,\mathbb{C}),\ \ r \geq 2$,
- $D_r \equiv \mathrm{so}(2r,\mathbb{C}),\ \ r \geq 4$,

and 5 "exceptional Lie algebras" (subscript = rank)

$$E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2.$$

Some comments:

- Including all $r \geq 1$, leads to redundancies:

$$A_1 \simeq B_1 \simeq C_1 \simeq D_1, \quad B_2 \simeq C_2, \quad D_2 \simeq A_1 \oplus A_1, \quad A_3 \simeq D_3. \tag{6.76}$$

- These Lie algebras, classified as complex Lie algebras over $\mathbb{C}$, have many different real forms over $\mathbb{R}$.

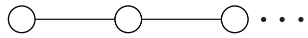
  Particularly important are the *compact real forms* in which

$$H^j = (H^j)^\dagger, \qquad E_{-\alpha} = (E_\alpha)^\dagger\,. \tag{6.77}$$

  $\hookrightarrow$ Relevant for the exponentiation to associated compact Lie groups!

Series of classical Lie algebras:

a) $A_r \equiv \mathrm{sl}(r+1,\mathbb{C}), \quad r \geq 1$

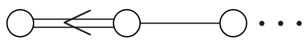


- Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}. \tag{6.78}$$

- compact real form: $A_r \to \mathrm{su(r+1)}, \quad r \geq 1.$

b) $B_r \equiv \mathrm{so}(2r+1,\mathbb{C}), \quad r \geq 3$

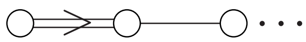


- Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & -2 \\ & & & -1 & 2 \end{pmatrix}. \tag{6.79}$$

- compact real form: $B_r \to \mathrm{so(2r+1)}, \quad r \geq 3.$

c) $C_r \equiv \mathrm{sp}(r,\mathbb{C}), \quad r \geq 2$

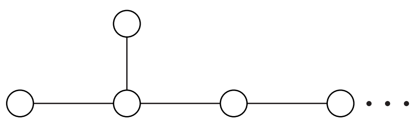


- Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & -1 \\ & & & -2 & 2 \end{pmatrix}. \tag{6.80}$$

- compact real form: $C_r \to \mathrm{usp(2r)}, \quad r \geq 2.$

d) $D_r \equiv \mathrm{so}(2r,\mathbb{C}), \quad r \geq 4$



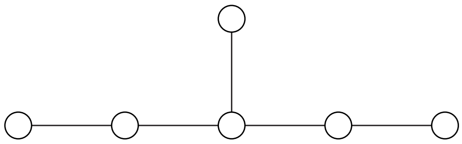
- Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & & & & \\ & & \ddots & -1 & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & \\ & & & -1 & & 2 \end{pmatrix}. \tag{6.81}$$

- compact real form: $D_r \to \mathrm{so(2r)}, \quad r \geq 4.$

Series of classical Lie algebras:

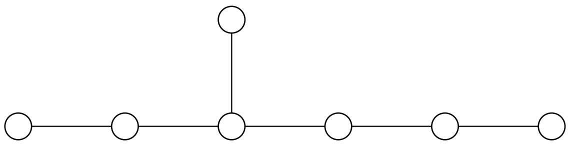
a) E_6



- Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & -1 & \\ & & -1 & 2 & & \\ & & -1 & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}. \tag{6.82}$$

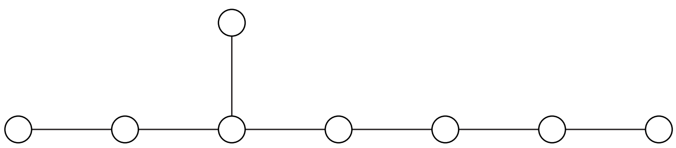
b) E_7



- Cartan matrix:

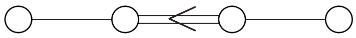
$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & \ddots & -1 & & & \\ & -1 & 2 & -1 & -1 & \\ & & -1 & 2 & & \\ & & -1 & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}. \tag{6.83}$$

c) E_8



- Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & \ddots & -1 & & & \\ & -1 & 2 & -1 & -1 & \\ & & -1 & 2 & & \\ & & -1 & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}. \tag{6.84}$$

d) F_4 

- Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -2 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}. \tag{6.85}$$

e) G_2 

- Cartan matrix:

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}. \tag{6.86}$$

6.5 Finite-dimensional representations of complex simple Lie algebras

6.5.1 Construction of irreducible weight systems

Preliminary considerations:

- $\mathcal{L}$ = complex simple Lie algebra with basis $\{H^i\}_{i=1}^r \cup \{E_\alpha\}_{\alpha\in\Phi}$ obeying $(H^i)^\dagger = H^i,\ E_{-\alpha} = (E_\alpha)^\dagger$.

  Recall: (compact real form of $\mathcal{L}$) = $\mathcal{L}_\mathrm{c} = \{X = X^\dagger \mid X \in \mathcal{L}\}$.

  ↪ Representations of $\mathcal{L}_\mathrm{c}$ exponentiate to *unitary* representations of corresponding compact Lie group G.

  ⇒ Finite-dim. representations of $\mathcal{L}$ determine $\underbrace{\text{finite-dim. } \textit{unitary} \text{ repr. of } G.}$

  ↪ Importance in QM and QFT!

- $\mathcal{L}$ = overlay of $\mathrm{sl}(2,\mathbb{C})$ algebras.

  ↪ Each representation R of $\mathcal{L}$ decomposes into several $\mathrm{sl}(2,\mathbb{C})$ representations.

  ↪ Make use of construction and properties of $\mathrm{sl}(2,\mathbb{C})$ representations!

Properties of finite-dim. representations R of $\mathcal{L}$:

- The repr. space V of R is spanned an orthonormal basis $\{v_k\}_{k=1}^{d_R}$, $\ d_R = \dim V < \infty$.
- All $R(H^i)$ are simultaneously diagonalizable.

  $\exists$ orthogonal subspaces $V_{(\lambda)}$ spanning $V = \oplus_\lambda V_{(\lambda)}$ with

$$R(H^i)\, v_{(\lambda)} = \lambda^i\, v_{(\lambda)} \qquad \forall v_{(\lambda)} \in V_{(\lambda)}, \qquad (\lambda) \equiv \big(\lambda^1, \ldots, \lambda^r\big) \tag{6.87}$$

  Each set $(\lambda) \neq 0$ defines a "weight" λ of R:

$$\lambda \equiv \lambda^i \Lambda_{(i)} \in \mathcal{H}^*. \tag{6.88}$$

  Notation for a generic "weight vector" $v_{(\lambda)} \in V_{(\lambda)}$:

$$|\lambda\rangle \equiv |\lambda^1, \ldots, \lambda^r\rangle \equiv v_{(\lambda)}. \tag{6.89}$$

$$\Rightarrow\ R(H^\alpha)\,|\lambda\rangle = R(\alpha_i H^i)\, v_{(\lambda)} = \alpha_i \lambda^i\, v_{(\lambda)} = (\alpha,\lambda)\,|\lambda\rangle \qquad \forall |\lambda\rangle = v_{(\lambda)} \in V_{(\lambda)}. \tag{6.90}$$

- Transition between different $V_{(\lambda)}$ via shift operators $E_{\pm\alpha}$:

$$\begin{aligned} R(H^\alpha)\,\big(R(E_{\pm\alpha})|\lambda\rangle\big) &= \big[R(H^\alpha),\, R(E_{\pm\alpha})\big]\,|\lambda\rangle + R(E_{\pm\alpha})\,R(H^\alpha)\,|\lambda\rangle \\ &= R\big([H^\alpha, E_{\pm\alpha}]\big)\,|\lambda\rangle + R(E_{\pm\alpha})\,(\alpha,\lambda)\,|\lambda\rangle \\ &= \pm(\alpha,\alpha)\,R(E_{\pm\alpha})\,|\lambda\rangle + (\alpha,\lambda)\,R(E_{\pm\alpha})\,|\lambda\rangle \\ &= (\alpha, \lambda\pm\alpha)\,\big(R(E_{\pm\alpha})|\lambda\rangle\big). \end{aligned} \tag{6.91}$$

  ⇒ For each weight λ, the states $R(E_{\pm\alpha})|\lambda\rangle$ are weight vectors $|\lambda\pm\alpha\rangle$ or zero.

- Each $\alpha \in \Phi$ defines some finite weight string through $|\lambda\rangle$: $\quad (p, q \in \mathbb{N}_0)$

$$|\lambda - p\alpha\rangle,\ |\lambda - (p-1)\alpha\rangle, \quad \dots, \quad |\lambda\rangle, \quad \dots, \quad |\lambda + q\alpha\rangle, \tag{6.92}$$

$$0 = R(E_{-\alpha})|\lambda - p\alpha\rangle, \qquad R(E_\alpha)|\lambda + q\alpha\rangle = 0. \tag{6.93}$$

From $\mathrm{sl}(2,\mathbb{C})$ representation theory:

$$(\alpha, \lambda - p\alpha) = -(\alpha, \lambda + q\alpha) \qquad \Rightarrow\ p - q = 2\frac{(\alpha,\lambda)}{(\alpha,\alpha)} = (\check{\alpha}, \lambda) \in \mathbb{Z}. \tag{6.94}$$

- Implications on components λ^i:
  Special case: $\quad \alpha = \alpha^{(i)}$ = simple root.

$$\mathbb{Z} \ni \left(\check{\alpha}^{(i)}, \lambda\right) = \lambda^i. \tag{6.95}$$

$\Rightarrow$ Weights $\lambda = \lambda^i \Lambda_{(i)}$ have integer components in Dynkin basis.

Weights λ with $\lambda^i \geq 0$ are called "dominant".

- R = finite-dim. $\quad \Rightarrow\ \exists$ highest weight Λ, i.e.

$$R(E_\alpha)|\Lambda\rangle = 0 \qquad \forall \alpha \in \Phi^+, \qquad (\Lambda) = (\Lambda^1, \dots, \Lambda^r), \quad \Lambda^i \in \mathbb{N}_0. \tag{6.96}$$

$\Rightarrow$ All $|\lambda\rangle$ can be obtained from some Λ according to

$$|\lambda\rangle = |\Lambda - \alpha - \beta \dots\rangle = R(E_{-\alpha})R(E_{-\beta})\dots|\Lambda\rangle. \tag{6.97}$$

Note: $\quad |\lambda\rangle = |\Lambda - (\text{some rows of } A)\rangle$, because components of $\alpha^{(i)}$ = ith row of A.

Highest-weight theorem:

For each dominant weight Λ there is a unique, irreducible, finite-dim. representation R_Λ of $\mathcal{L}$, and each irreducible, finite-dim. representation corresponds to a dominant weight.

Algorithm for determining all weights of R_Λ:

1. Weight of "level 0" = given highest weight Λ with integer $\Lambda^i \geq 0$.
2. Weights of "level 1":
   a) Apply $R(E_{-\alpha^{(i)}})$ for all pos. simple roots $\alpha^{(i)} \in \Phi^+$ to $|\Lambda\rangle$.
   b) Calculate the new potential root $|\lambda\rangle = |\Lambda - (i\text{th row of } A)\rangle$.
   c) Check $p = q + \Lambda^i > 0$ with (6.94), i.e. whether $|\lambda - \alpha^{(i)}\rangle$ is still in the weight string. (At this level, $q = 0\ \forall i$.)
3. Weights of "level 2" and higher: $\quad$ Iterate step 2!
   a) Subtract each row of A from each $|\lambda\rangle$ of the previous level.
   b) Check $p = q + \lambda^i > 0$ with (6.94), i.e. whether each new potential weight $|\lambda - \alpha^{(i)}\rangle$ is still in the weight string. (q is the largest integer with $|\lambda + q\alpha^{(i)}\rangle$ being a weight of lower level.)

   Repeat this step until no more weights are obtained.

Comment: The algorithm does not determine the multiplicity of weight vectors $|\lambda\rangle$.
$\hookrightarrow$ Done later (see Section 6.5.3)!

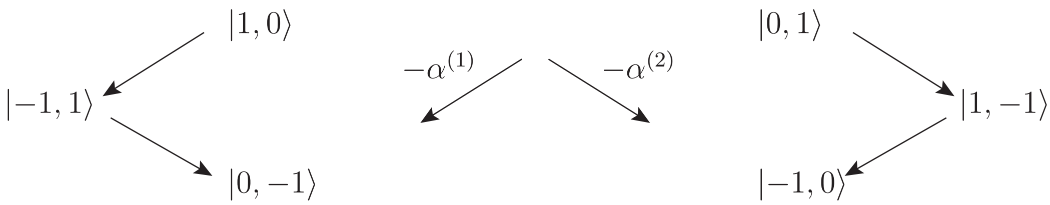
Specific representations:

- "Fundamental representations" = representations with the fundamental weights $\Lambda_{(i)}$ as highest weight, i.e. in components $(\Lambda) = (1, 0, \dots), (0, 1, 0, \dots), \dots$.
- Adjoint representation R_{ad}: roots $\equiv$ weights of R_{ad}.

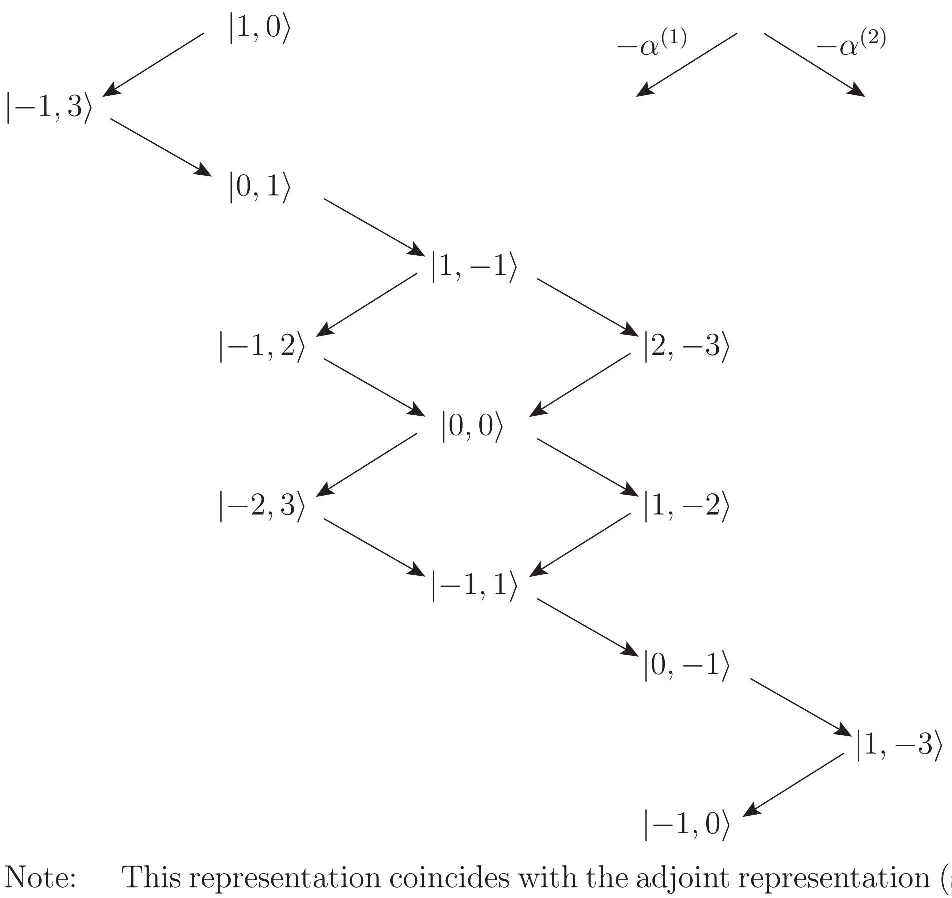
  Highest weight Λ_{ad} = maximal root θ = unique, and all $\Lambda^i_{\text{ad}} > 0$.

Examples:

- Fundamental representations of $\text{sl}(3, \mathbb{C}) = A_2$, $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.



- Fundamental representation of G_2 for $|\Lambda\rangle = |1, 0\rangle$, $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.



Note: This representation coincides with the adjoint representation (see Section 6.4).

6.5.2 Quadratic Casimir operator and index of a representation

Recall:

- Def.: $\mathcal{C}$ = Casimir operator in some representation R of $\mathcal{L}$.
  $\Leftrightarrow$ $[\mathcal{C}, R(x)] = 0 \ \ \forall x \in \mathcal{L}$.
- Schur's lemma: R = irreducible $\Rightarrow$ $\mathcal{C} = C_R \cdot \mathbb{1}_{d_R}$.
  $\Rightarrow$ Casimir operators characterize representations.

Quadratic Casimir operator:

If $\mathcal{L}$ is a semisimple Lie algebra generated by $\{T^A\}_{A=1}^{d_\mathcal{L}}$, then

$$\mathcal{C} = g_{AB}\, T^A T^B \tag{6.98}$$

is a Casimir operator.

Note: Evaluating $\mathcal{C}$ actually requires to go into some representation, because T^AT^B in general is undefined in $\mathcal{L}$.

Proof:

$$\begin{aligned}
\left[T^C, \mathcal{C}\right] &= g_{AB}\left[T^C, T^AT^B\right] = g_{AB}\Big(\underbrace{\left[T^C, T^A\right]}_{=\mathrm{i}f^{CA}{}_D T^D} T^B + T^A \underbrace{\left[T^C, T^B\right]}_{=\mathrm{i}f^{CB}{}_D T^D}\Big)\\
&= \mathrm{i} g_{AB}\, f^{CA}{}_D \left(T^DT^B + T^BT^D\right) \qquad \text{using symmetry } A \leftrightarrow B \text{ in 2nd term}\\
&= \mathrm{i} g_{AB}\, g_{DE}\, f^{CAE} \left(T^DT^B + T^BT^D\right)\\
&= \frac{\mathrm{i}}{2}\left(g_{AB}\, g_{DE} + g_{AD}\, g_{BE}\right)\, f^{CAE} \left(T^DT^B + T^BT^D\right) \qquad \text{using symmetry } B \leftrightarrow D\\
&= \frac{\mathrm{i}}{2} g_{AB}\, g_{DE} \big(\underbrace{f^{CAE} + f^{CEA}}_{=\ 0 \text{ due to antisymmetry of } f^{CAE},\ \text{cf. (5.68)}}\big) \left(T^DT^B + T^BT^D\right) \qquad \text{renaming } A \leftrightarrow E\\
&= 0.
\end{aligned}$$

#

$\mathcal{C}$ in Cartan–Weyl basis $\{H^i\}_{i=1}^r \cup \{E_\alpha\}_{\alpha\in\Phi}$:

$$\mathcal{C} = g_{ij} H^i H^j + \sum_{\alpha\in\Phi} E_\alpha E_{-\alpha}, \qquad \text{if } (E_\alpha, E_{-\alpha}) = 1. \tag{6.99}$$

Proof:

This is a consequence of the block structure of the Killing form (g^{AB}):

$$(g^{AB}) = \left(\begin{array}{c|ccc} (g^{ij}) & & 0 & \\ \hline & \sigma_1 & & \\ 0 & & \sigma_1 & \\ & & & \ddots \end{array}\right), \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

#

Calculation of C_R is representation R_Λ: Use $E_\alpha|\Lambda\rangle = 0 \;\; \forall\alpha \in \Phi^+$.

$$\mathcal{C}\,|\Lambda\rangle = \Big(g_{ij}H^i H^j + \sum_{\alpha\in\Phi} E_\alpha E_{-\alpha}\Big)|\Lambda\rangle = \Big(g_{ij}\Lambda^i\Lambda^j + \sum_{\alpha\in\Phi^+} \underbrace{[E_\alpha, E_{-\alpha}]}_{=H^\alpha}\Big)|\Lambda\rangle$$
$$= \Big((\Lambda,\Lambda) + \sum_{\alpha\in\Phi^+}(\Lambda,\alpha)\Big)|\Lambda\rangle.$$

Defining

$$\rho \equiv \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha = \text{“Weyl vector”}, \tag{6.100}$$

this yields

$$\mathcal{C}\,|\Lambda\rangle = (\Lambda, \Lambda+2\rho)\,|\Lambda\rangle, \qquad C_R = (\Lambda, \Lambda+2\rho)\,/\,d_R. \tag{6.101}$$

Index of a representation *R*:

A statement about invariant bilinear forms on $\mathcal{L}$:

For a simple Lie algebra $\mathcal{L}$, any invariant bilinear form $(x,y)'$ differs by the Killing form $(x,y) = \mathrm{Tr}(\mathrm{ad}_x, \mathrm{ad}_y)$ only by a constant factor.

Proof: Exercise?! (See also Ref. [1].)

$\hookrightarrow$ Definition: The “index” I_R of a repr. R with generators $\{T_R^A\}_{A=1}^{d_\mathcal{L}}$ is defined by

$$\mathrm{Tr}\left(T_R^A\, T_R^B\right) = I_R \cdot g^{AB}. \tag{6.102}$$

Connection between I_R and C_R:

$$\begin{aligned}\mathrm{Tr}_{\mathrm{ad}}(\mathcal{C}) &= g_{AB}\,\mathrm{Tr}\left(T_{\mathrm{ad}}^A\,T_{\mathrm{ad}}^B\right) = g_{AB}\,g^{AB} = d_\mathcal{L},\\ \mathrm{Tr}_R(\mathcal{C}) &= g_{AB}\,\mathrm{Tr}\left(T_R^A\,T_R^B\right) = I_R\cdot g_{AB}\,g^{AB} = I_R\,d_\mathcal{L},\\ &= C_R\,d_R. \end{aligned}\tag{6.103}$$

$$\Rightarrow\; I_R = \frac{d_R}{d_\mathcal{L}}\,C_R = \frac{d_R}{d_\mathcal{L}}\,(\Lambda, \Lambda+2\rho). \tag{6.104}$$

6.5.3 Multiplets of irreducible representations – Freudenthal's formula

Goal: Complete algorithm of Section 6.5.1 by determining the multiplicity $n_\lambda = \dim V_{(\lambda)}$ of each weight vector $|\lambda\rangle$.

Idea: Calculate $\mathrm{Tr}(\mathcal{C})$ restricted to subspace $V_{(\lambda)}$ in two different ways.
$\hookrightarrow$ Recursion relation for n_λ.

1. Use result for C_R:

$$\mathrm{Tr}_R(\mathcal{C})\big|_{V_{(\lambda)}} = C_R\, n_\lambda = (\Lambda, \Lambda + 2\rho)\, n_\lambda. \tag{6.105}$$

2. Use general form of $\mathcal{C}$:

$$\mathrm{Tr}_R(\mathcal{C})\big|_{V_{(\lambda)}} = \mathrm{Tr}_R\Big(g_{ij} H^i H^j + \sum_{\alpha\in\Phi} E_\alpha E_{-\alpha}\Big)\Big|_{V_{(\lambda)}}. \tag{6.106}$$

Evaluation of 1st part with basis $\{|\lambda; l\rangle\}_{l=1}^{n_\lambda}$ of $V_{(\lambda)}$:

$$\begin{aligned}\mathrm{Tr}_R\left(g_{ij}H^iH^j\right)\big|_{V_{(\lambda)}} &= \sum_{l=1}^{n_\lambda} g_{ij}\,\langle\lambda; l|H^iH^j|\lambda; l\rangle = \sum_{l=1}^{n_\lambda} g_{ij}\,\lambda^i\lambda^j\,\underbrace{\langle\lambda; l|\lambda; l\rangle}_{=1}\\ &= n_\lambda\,(\lambda,\lambda).\end{aligned} \tag{6.107}$$

3. Evaluation of 2nd part of (6.106) via $\mathrm{sl}(2,\mathbb{C})$ weight strings:
Each α-string corresponds to a multiplet of eigenstates $|t, t_3\rangle$ with t = fixed and

$$\begin{aligned}\vec{T}^{\,2}\,|t,t_3\rangle &= t(t+1)\,|t,t_3\rangle,\\ T_3\,|t,t_3\rangle &= t_3\,|t,t_3\rangle, \qquad t_3 = -t, -t+1, \ldots, t.\end{aligned} \tag{6.108}$$

Relation between $\vec{T}^{\,2}$, T_a and H^α, $E_{\pm\alpha}$ ($\alpha > 0$), cf. (6.49):

$$\begin{aligned}&T_3 = \frac{1}{2}h_\alpha = \frac{H^\alpha}{(\alpha,\alpha)}, \qquad T_\pm = e_{\pm\alpha} = \sqrt{\frac{2}{(\alpha,\alpha)}}\,E_{\pm\alpha},\\ &[T_3, T_\pm] = \frac{1}{2}\,[h_\alpha, e_{\pm\alpha}] = \pm\, e_{\pm\alpha} = \pm T_\pm, \quad [T_+, T_-] = [e_\alpha, e_{-\alpha}] = h_\alpha = 2T_3.\\ &\Rightarrow\ \vec{T}^{\,2} = T_3^2 + \frac{1}{2}\,(T_+T_- + T_-T_+) = \frac{(H^\alpha)^2}{(\alpha,\alpha)^2} + \frac{E_\alpha E_{-\alpha} + E_{-\alpha}E_\alpha}{(\alpha,\alpha)}.\end{aligned} \tag{6.109}$$

Since $\vec{T}^{\,2} = t(t+1)$ on the weight string, we get

$$E_\alpha E_{-\alpha} + E_{-\alpha}E_\alpha = t(t+1)\,(\alpha,\alpha) - \frac{(H^\alpha)^2}{(\alpha,\alpha)}. \tag{6.110}$$

Identify the state $|t, t_3 = t\rangle$ with the highest-weight state $|\lambda + k\alpha\rangle$ of the string:

$$t\,|t,t\rangle = T_3\,|t,t\rangle = \frac{H^\alpha}{(\alpha,\alpha)}\,|\lambda + k\alpha\rangle = \frac{(\alpha, \lambda + k\alpha)}{(\alpha,\alpha)}\,|\lambda + k\alpha\rangle. \qquad \Rightarrow\ t = \frac{(\alpha,\lambda + k\alpha)}{(\alpha,\alpha)}.$$

$\Rightarrow$ Application of $E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha$ to basis state $|\lambda; l\rangle \in V_{(\lambda)}$:

$$\begin{aligned}(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)\,|\lambda; l\rangle &= \left(t(t+1)\,(\alpha,\alpha) - \frac{(H^\alpha)^2}{(\alpha,\alpha)} \right) |\lambda; l\rangle \\ &= \left(t(t+1)\,(\alpha,\alpha) - \frac{(\alpha,\lambda)^2}{(\alpha,\alpha)} \right) |\lambda; l\rangle \\ &= \big(k(k+1)\,(\alpha,\alpha) + (2k+1)\,(\alpha,\lambda)\big)\,|\lambda; l\rangle. \qquad (6.111)\end{aligned}$$

Note: k-value depends on l, $k = k_l$, i.e. k differs for different $|\lambda; l\rangle$:

- $n_\lambda = (\#\text{ states } |\lambda; l\rangle \text{ with arbitrary } k)$,
- $n_\lambda - n_{\lambda+\alpha} = (\#\text{ states } |\lambda; l\rangle \text{ with } k = 0)$,
- $n_{\lambda+k\alpha} - n_{\lambda+(k+1)\alpha} = (\#\text{ states } |\lambda; l\rangle \text{ for given } k = k_l)$,
- $n_{\lambda+k\alpha} = 0$ for sufficiently large k.

$\Rightarrow$ $\sum_{l=1}^{n_\lambda} f(k_l) = \sum_{k=0}^{\infty} (n_{\lambda+k\alpha} - n_{\lambda+(k+1)\alpha}) f(k)$

Evaluation of remaining part of $\mathrm{Tr}_R(\mathcal{C})|_{V_{(\lambda)}}$:

$$\begin{aligned}\mathrm{Tr}_R\Big(\sum_{\alpha\in\Phi} E_\alpha E_{-\alpha}\Big)\Big|_{V_{(\lambda)}} &= \sum_{\alpha\in\Phi^+} \mathrm{Tr}_R\,(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)\,\big|_{V_{(\lambda)}} \\ &= \sum_{\alpha\in\Phi^+} \sum_{l=1}^{n_\lambda} \langle \lambda; l|\, E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha\, |\lambda; l\rangle \\ &= \sum_{\alpha\in\Phi^+} \sum_{k=0}^{\infty} (n_{\lambda+k\alpha} - n_{\lambda+(k+1)\alpha}) \Big(k(k+1)\,(\alpha,\alpha) + (2k+1)\,(\alpha,\lambda) \Big) \\ &= \sum_{\alpha\in\Phi^+} \sum_{k=0}^{\infty} n_{\lambda+k\alpha} \Big(k(k+1)\,(\alpha,\alpha) + (2k+1)\,(\alpha,\lambda) \Big) \\ &\quad - \sum_{\alpha\in\Phi^+} \sum_{k=1}^{\infty} n_{\lambda+(k+1)\alpha} \Big((k-1)k\,(\alpha,\alpha) + (2k-1)\,(\alpha,\lambda) \Big) \\ &= n_\lambda \sum_{\alpha\in\Phi^+} (\alpha,\lambda) + \sum_{\alpha\in\Phi^+} \sum_{k=1}^{\infty} n_{\lambda+k\alpha} \Big(2k\,(\alpha,\alpha) + 2\,(\alpha,\lambda) \Big) \\ &= n_\lambda\,(2\rho,\lambda) + 2 \sum_{\alpha\in\Phi^+} \sum_{k=1}^{\infty} n_{\lambda+k\alpha}\,(\alpha, \lambda + k\alpha). \qquad (6.112)\end{aligned}$$

4. Final relation upon combining (6.105), (6.107), and (6.112):

$$n_\lambda = \frac{2 \sum_{\alpha\in\Phi^+} \sum_{k=1}^{\infty} n_{\lambda+k\alpha}\,(\alpha, \lambda + k\alpha)}{(\Lambda - \lambda, \Lambda + \lambda + 2\rho)}. \qquad \text{("Freudenthal's formula")} \qquad (6.113)$$

Algorithm to determine n_λ for known weights λ:

- Proceed recursively in increasing level of λ, starting with level 0: $n_\Lambda = 1$.
  $\hookrightarrow$ R.h.s. of (6.113) can be assumed to be known.
- Evaluation of denominator of (6.113):
  - Expand $(\Lambda - \lambda)$ in terms of simple roots: $\Lambda - \lambda = c_i \alpha^{(i)}$.
  - Represent $(\Lambda + \lambda + 2\rho)$ in Dynkin basis: $\Lambda + \lambda + 2\rho = d^i \Lambda_{(i)}$.
    Use non-trivial relation for ρ: $\rho = \Lambda_{(i)}$.

  $$\Rightarrow\ (\Lambda - \lambda, \Lambda + \lambda + 2\rho) = c_i\, d^j \underbrace{\left(\alpha^{(i)}, \Lambda_{(j)}\right)}_{=\frac{1}{2}\left(\alpha^{(i)},\alpha^{(i)}\right)\delta^i_j} = \tfrac{1}{2}\sum_{i=1}^r c_i\, d_i\, \left(\alpha^{(i)}, \alpha^{(i)}\right).$$
- Evaluation of numerator of (6.113):
  - $n_{\lambda+k\alpha}$ known from previous steps.
  - $(\alpha, \lambda + k\alpha)$ calculable via (6.94):
    $(\alpha, \lambda + k\alpha) = k\,(\alpha,\alpha) + (\alpha,\lambda) = \left(k + \frac{1}{2}(p-q)\right)(\alpha,\alpha)$,
    after reading p, q from weight diagram.
- Simple cases:
  $n_\lambda = 1$ if there is only one possibility to come to $|\lambda\rangle$ via $E_{-\alpha}E_{-\beta}\cdots|\Lambda\rangle$ with $\alpha, \beta > 0$ (or via $E_\alpha E_\beta \cdots |\Lambda_{\min}\rangle$).

Example of Section 6.5.1 reloaded: G_2 representation with $|\Lambda\rangle = |1,0\rangle$.

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \qquad \left(\alpha^{(1)},\alpha^{(1)}\right) = 3, \qquad \left(\alpha^{(2)},\alpha^{(2)}\right) \equiv 1, \qquad \left(\alpha^{(1)},\alpha^{(2)}\right) = -\frac{3}{2}.$$

- $n_\lambda = 1$ obvious for all $|\lambda\rangle \neq |0,0\rangle$.
- $|\lambda\rangle = |0,0\rangle$:
  Denominator:

$$\Lambda - \lambda = \Lambda = 2\alpha^{(1)} + 3\alpha^{(2)},$$
$$(\Lambda + \lambda + 2\rho) = (1,0) + (0,0) + 2\cdot(1,1) = (3,2),$$
$$\Rightarrow\ (\Lambda - \lambda, \Lambda + \lambda + 2\rho) = \frac{1}{2}\,(2\cdot 3\cdot 3 + 2\cdot 3)\left(\alpha^{(2)},\alpha^{(2)}\right) = 12.$$

6 numerator contributions from 6 positive roots α:

$k\alpha$	k	p	q	(α,α)	$2n_{\lambda+k\alpha}\,(\alpha,\lambda+k\alpha)$
$\alpha^{(1)}$	1	1	1	3	6
$\alpha^{(2)}$	1	1	1	1	2
$\alpha^{(1)}+\alpha^{(2)}$	1	1	1	1	2
$\alpha^{(1)}+2\alpha^{(2)}$	1	1	1	1	2
$\alpha^{(1)}+3\alpha^{(2)}$	1	1	1	3	6
$2\alpha^{(1)}+3\alpha^{(2)}$	1	1	1	3	6
sum:					24

$\Rightarrow\ n_{(0,0)} = \frac{24}{12} = 2.$

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