Exercises to Group Theory for Physicists — Sheet 11

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Exercise 11.1 *Baker–Campbell–Hausdorff (BCH) formula* (3 points)

The general BCH formula for two sufficiently small elements X, Y in a Lie algebra $\mathcal{L}$ reads:

$$-\mathrm{i}\ln\left(\mathrm{e}^{\mathrm{i}X}\,\mathrm{e}^{\mathrm{i}Y}\right) = X + \int_0^1 \mathrm{d}t\, g\left(e^{\mathrm{i}\,\mathrm{ad}_X}\, e^{\mathrm{i}t\,\mathrm{ad}_Y}\right)(Y), \qquad g(z) \equiv \frac{\ln z}{1-1/z}. \tag{1}$$

a) Derive the differential form

$$\ln\left(\mathrm{e}^{\mathrm{i}X}\,\mathrm{e}^{\mathrm{i}Y}\right) = \mathrm{i}X + \mathrm{i}Y - \frac{1}{2}[X,Y] - \frac{\mathrm{i}}{12}[X,[X,Y]] + \frac{\mathrm{i}}{12}[Y,[X,Y]] + \ldots, \tag{2}$$

of the BCH formula (up to the terms included here) starting from (1).

b) Suppose that X, Y are upper triangular matrices with zeroes on the diagonal. What happens to the differential form (2) of the BCH formula in this case?

Exercise 11.2 *Example for a non-compact Lie group* (6 points)

Consider the matrix Lie group defined by the following matrices

$$A(\theta_1, \theta_2) = \begin{pmatrix} \theta_1 & \theta_2 \\ 0 & 1 \end{pmatrix}, \tag{3}$$

where the group parameters θ_1, θ_2 are restructed to $\theta_1 \in \mathbb{R}^+$, $\theta_2 \in \mathbb{R}$.

a) Derive the generators T^A of the Lie group and the structure constants of the group and describe the corresponding Lie algebra. Are there invariant Lie subgroups and invariant Lie subalgebras?

b) Calculate the exponentiation of a general element $\tau_A T^A$, $\tau_1, \tau_2 \in \mathbb{R}$, of the Lie algebra. Use this to calculate the group element $\exp\{-\mathrm{i}\tau_A T^A\}$. Is every group element A expressible in terms of such an exponential? Answer this question also for the extension of the group to $\theta_1 < 0$.

c) Calculate the left-invariant group measure $\mathrm{d}\mu_\mathrm{L}(A)$ as described in the lecture. Calculate the right-invariant group measure $\mathrm{d}\mu_\mathrm{R}(A)$ following an analogous reasoning. Do left and right measures agree?