Exercises to Group Theory for Physicists — Sheet 10

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Exercise 10.1 *Lie algebras of dimension 2* (2 points)

Consider the most general Lie algebra spanned by the two generators X_1 and X_2.

a) Show that this Lie algebra can be brought to the canonical form

$$[X_1', X_2'] = X_1' \tag{1}$$

by a linear transformation of the generators.

b) Which conclusion can you draw from a) on the simplicity and semisimplicity of 2-dimensional Lie algebras?

Exercise 10.2 *Lorentz group in 2+1 dimensions* (5 points)

The homogenous Lorentz group in 2+1 dimensions, called O(1,2), is defined by all linear transformations represented by some real 3×3 matrix $L(\phi, \nu_1, \nu_2)$ preserving the "Minkowskian length"

$$l(\vec{x})^2 \equiv x_0^2 - x_1^2 - x_2^2 \tag{2}$$

of some vector $\vec{x} = (x_0, x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^3$. In the following we consider the identity component of O(1,2), called $\mathrm{SO}^+(1,2)$.

a) Show that $L(\phi, \nu_1, \nu_2)$ can be written near the identity as

$$L(\delta\phi, \delta\nu_1, \delta\nu_2) = \begin{pmatrix} 1 & \delta\nu_1 & \delta\nu_2 \\ \delta\nu_1 & 1 & -\delta\phi \\ \delta\nu_2 & \delta\phi & 1 \end{pmatrix} \equiv \mathbb{1} - \mathrm{i}\delta\phi J_3 - \mathrm{i}\delta\nu_1 K_1 - \mathrm{i}\delta\nu_2 K_2 \tag{3}$$

with infinitesimal group parameters $\delta\phi, \delta\nu_1, \delta\nu_2$ and the matrix-valued group generators J_3, K_1, K_2. What is the physical meaning of the group parameters?

b) Derive the explicit form of the matrices $L_3(\phi) = L(\phi, 0, 0)$, $L_1(0, \nu_1, 0)$, and $L_2(0, 0, \nu_2)$ of the one-parameter subgroups corresponding to the generators J_3, K_1, K_2.

c) Derive the canonical commutators of the Lie algebra spanned by J_3, K_1, K_2. Give a geometrical interpretation of the result for $[K_1, K_2]$.

d) Associating to each vector $\vec{x}$ the hermitian matrix

$$X = x_0\,\mathbb{1} - x_1\,\sigma_1 - x_2\,\sigma_2 = \begin{pmatrix} x_0 & -x_1 + \mathrm{i}x_2 \\ -x_1 - \mathrm{i}x_2 & x_0 \end{pmatrix}, \quad \sigma_k = \text{Pauli matrices},$$

show that the transformations $X \mapsto X' = A\,X\,A^{\mathrm{T}}$ provide another representation of $\mathrm{SO}^+(1,2)$ if A is a real 2×2 matrix with $\det A = \pm 1$.

e) What is the group-theoretical relation between $\mathrm{SO}^+(1,2)$ and $\mathrm{SL}(2,\mathbb{R})$?
Hint: Recall the relation between SO(3) and SU(2) and argue along the same line.

Please turn over!

Exercise 10.3 *An operator identity* (2 points)

Prove the identity

$$\exp(A)\, B\, \exp(-A) = \exp(\mathrm{ad}_A)(B), \tag{4}$$

where A and B are linear operators and

$$(\mathrm{ad}_A)^k(B) \equiv \underbrace{[A, [\ldots, [A, B], \ldots]]}_{k \text{ commutators}}. \tag{5}$$

Hint: Show first the auxiliary identity

$$(\mathrm{ad}_A)^k(B) = \sum_{l=1}^{k} \binom{k}{l} A^l\, B\, (-A)^{k-l}. \tag{6}$$