Exercises to Group Theory for Physicists — Sheet 6
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Exercise 6.1 *Real, pseudoreal, and complex representations of Lie groups* (3 points)

Let D be a finite-dim. representation of a Lie group G of dimensions n and

$$D(\theta_1, \dots, \theta_n) \;=\; \exp\{-\mathrm{i}T^a\theta_a\} \tag{1}$$

be the corresponding representation matrices, where θ_a and T^a are the group parameters and generators (in D represenation), respectively.

a) Translate the condition on D for being a real, pseudoreal, or complex representation into a condition on the generators T^a.

b) Is the 3-dim. defining representation of SO(3) real, pseudoreal, or complex? If (pseudo)real, give the bilinear invariant $(x, y) = x^{\mathrm{T}}Sy$, where $x, y \in \mathbb{R}^3$.

c) Is the 2-dim. defining representation of SU(2) real, pseudoreal, or complex? If (pseudo)real, give the bilinear invariant $\langle x, y\rangle = x^{\mathrm{T}}Sy$, where $x, y \in \mathbb{R}^2$.

Exercise 6.2 *Characters of irreducible SU(2) representations* (4 points)

Consider Wigner's $D^{(j)}$ functions which are defined by

$$\begin{aligned} D^{(j)}(\alpha,\beta,\gamma)_{m'm} &= \langle j,m'|\,\exp\Big\{-\mathrm{i}\alpha\,J_3^{(j)}\Big\}\,\exp\Big\{-\mathrm{i}\beta\,J_2^{(j)}\Big\}\,\exp\Big\{-\mathrm{i}\gamma\,J_3^{(j)}\Big\}\,|j,m\rangle \\ &= \mathrm{e}^{-\mathrm{i}m'\alpha-\mathrm{i}m\gamma}\,\langle j,m'|\,\exp\Big\{-\mathrm{i}\beta\,J_2\Big\}\,|j,m\rangle = \mathrm{e}^{-\mathrm{i}m'\alpha-\mathrm{i}m\gamma}\,d^{(j)}_{m'm}(\beta), \end{aligned} \tag{2}$$

where $\vec{J}^{(j)}$ is the angular momentum operator in the $(2j + 1)$-dim. spin-j representation and α, β, and γ are the usual Euler angles as defined in the lecture.

a) Prove that the characters $\chi^{(j)} \equiv \mathrm{Tr}\{D^{(j)}\}$ of the spin-j representation are given by

$$\chi^{(j)}(\alpha,\beta,\gamma) \equiv \chi^{(j)}(\theta) = \frac{\sin\Big(\theta(j+\frac{1}{2})\Big)}{\sin(\frac{\theta}{2})}, \tag{3}$$

where θ is the angle of the single rotation around some axis $\vec{e}$ described by the three Euler rotations, i.e. $D^{(j)}(\alpha,\beta,\gamma) = D^{(j)}(\theta\vec{e}) = \exp\{-\mathrm{i}\theta\vec{e}\cdot\vec{J}^{(j)}\}$.

b) Prove the following orthogonality relation by direct integration,

$$\int_0^{2\pi}\mathrm{d}\alpha\int_0^{\pi}\mathrm{d}\beta\,\sin\beta\int_0^{2\pi}\mathrm{d}\gamma\,\chi^{(j_1)}(\alpha,\beta,\gamma)^*\,\chi^{(j_2)}(\alpha,\beta,\gamma) = 8\pi^2\,\delta_{j_1j_2}, \tag{4}$$

using the relation between the angles α, β, γ and θ given in the lecture,

$$\cos\theta = \cos\beta\,\cos^2\left(\frac{\alpha+\gamma}{2}\right) - \sin^2\left(\frac{\alpha+\gamma}{2}\right). \tag{5}$$

Please turn over!

Exercise 6.3 *Recursion relation for Clebsch–Gordan coefficients* (2 points)

We consider a quantum-mechanical system consisting of two parts that are each described by angular momentum eigenstates $|j_k, m_k\rangle$ ($k = 1, 2$) of $\vec{J}_k^2$ and $J_{k,3}$ of the respective angular momentum operators $\vec{J}_k$:

$$\vec{J}_k^2 |j_k, m_k\rangle = \hbar^2 j_k(j_k+1)|j_k, m_k\rangle, \qquad j_k = 0, \tfrac{1}{2}, 1, \ldots,$$
$$J_{k,3}|j_k, m_k\rangle = \hbar m_k |j_k, m_k\rangle, \qquad m_k = -j_k, -j_k+1, \ldots, j_k.$$

The transition from the basis of product states $|j_1, m_1; j_2, m_2\rangle \equiv |j_1, m_1\rangle|j_2, m_2\rangle$ to the basis $|j, m\rangle$ of eigenstates of $\vec{J}^2$ and J_3 of the total angular momentum $\vec{J}$ is described in terms of Clebsch–Gordan coefficients $\langle j_1, m_1; j_2, m_2|j, m\rangle$:

$$|j, m\rangle = \sum_{\substack{m_1, m_2 \\ m = m_1 + m_2}} |j_1, m_1; j_2, m_2\rangle \, \langle j_1, m_1; j_2, m_2|j, m\rangle. \tag{6}$$

With the help of the shift operators $J_\pm = J_{1\pm} + J_{2\pm}$ derive the following recursion relations for the Clebsch–Gordan coefficients:

$$\begin{aligned} \sqrt{j(j+1) - m(m-1)} \, &\langle j_1, m_1; j_2, m_2|j, m-1\rangle \\ &= \sqrt{j_1(j_1+1) - m_1(m_1+1)} \, \langle j_1, m_1+1; j_2, m_2|j, m\rangle \\ &+ \sqrt{j_2(j_2+1) - m_2(m_2+1)} \, \langle j_1, m_1; j_2, m_2+1|j, m\rangle. \end{aligned}$$