Exercises to Group Theory for Physicists — Sheet 5

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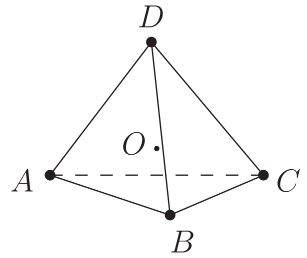
Exercise 5.1 *Tetrahedral symmetry group T* (5 points)

The tetrahedral symmetry group T is the group of all rotational symmetries of a tetrahedron (i.e. without reflexions), comprising 12 elements: the identity e, two rotations (with angles 120° and 240°) around each of the four axes OA, OB, OC, OD, and three rotations (with angle 180°) around the three axes linking opposite edges (e.g. AD and BC).



a) The tetrahedral group can be defined by the presentation $\langle a, b \,|\, a^2 = b^3 = (ab)^3 = e\rangle$. Show that

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

generate a 3-dimensional representation of the group. Is this representation irreducible or reducible? (Prove your answer!)

b) What are the dimensions of all inequivalent irreducible representations of T?

c) What are the classes of T?

d) Determine the character table of T.

e) Which statement can be made on the possible degrees of degeneracy of energy eigenstates of an electron in a molecule that has T as its symmetry group?

Exercise 5.2 *Rotation matrices* (2 points)

Consider a rotation about the vector $\vec{\theta} = \theta\vec{e}$ in 3-dimensional space, i.e. about an axis $\vec{e}$ ($\vec{e}^2 = 1$) with an angle θ ($0 \leq \theta \leq \pi$).

a) Show that the 3×3 matrix $R(\vec{\theta})$ for this rotation is given by

$$R(\vec{\theta}) = \cos\theta\,\mathbb{1} + (1-\cos\theta)\,\vec{e}\,\vec{e}^{\mathrm{T}} + \sin\theta\,\vec{e}\cdot(-\mathrm{i}\vec{I}\,), \tag{1}$$

upon directly evaluating the exponential series $R(\vec{\theta}) = \exp\{-\mathrm{i}\vec{\theta}\cdot\vec{I}\}$ with $(I_a)_{bc} = -\mathrm{i}\epsilon_{abc}$.

b) Derive formulas that deliver θ and the components of $\vec{e}$ directly from the components of the matrix $R(\vec{\theta})$. Use these results to determine θ and $\vec{e}$ for the rotation

$$R(\vec{\theta}) = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}. \tag{2}$$

Please turn over!

Exercise 5.3 *Euler's ϕ-function and the modular inverse* (2 points)

Let G_p, $p \in \mathbb{N}$, $p > 1$, be the set of integers $1 \leq k < p$ which are co-prime to p (i.e. the only common factor of k and p is 1). The multiplication modulo p, denoted by $\circ$, is obviously associative and has the neutral element 1.

a) Show that G_p is closed w.r.t. $\circ$, i.e. that $a \circ b \in G_p \; \forall \, a, b \in G_p$.

   Hint: ab and p are co-prime by assumption. Use a proof by contradiction.

b) For k, p co-prime it is guaranteed that the equation $xk + yp = 1$ has a solution for $x, y \in \mathbb{Z}$, i.e. the modular inverse $x = k^{-1} \pmod{p}$ exists. Show that $k^{\phi(p)} = 1 \pmod{p} \; \forall \, k \in G_p$ (Euler's theorem), where $\phi(p) = |G_p|$ is Euler's ϕ-function. Use this to find the inverse of k. What is $\phi(p)$ in the case that p is prime?

Remarks:

- The modular inverse can also be calculated by the extended Euclidean algorithm.
- For p prime, the set $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is a field w.r.t. addition $+$ and multiplication $\circ$ both modulo p, i.e. $(\mathbb{F}_p, +)$ is an abelian group with neutral element 0, $(\mathbb{F}_p \backslash \{0\}, \circ)$ is an abelian group, and distributive laws hold. The so-called *modular arithmetic* on $\mathbb{F}_p$ is of fundamental importance in modern computer algebra.