Exercise 2.1 *Three-dimensional harmonic oscillator* (3 points)

The Hamilton operator of the three-dimensional isotropic oscillator is

$$\hat{H} = \frac{\hat{\vec{p}}^{\,2}}{2m} + \frac{m\omega^2}{2}\,\hat{\vec{x}}^{\,2}, \tag{1}$$

with $\hat{\vec{x}}$ and $\hat{\vec{p}}$ denoting the cartesian position and momentum operators of a particle of mass m, respectively.

a) Which are the conserved quantities of the system?

b) Using your knowledge about the one-dimensional harmonic oscillator and its eigenstates, determine the energy spectrum and the corresponding eigenstates of the three-dimensional system by separating the cartesian variables.

c) What is the degree of degeneracy of the energy levels?

Exercise 2.2 *Simplified version of Wigner's theorem* (2 points)

a) Show that a *linear*, invertible operator U that preserves all norms of states in a Hilbert space, i.e.

$$||\psi|| = ||U\psi|| \qquad \text{for all } |\psi\rangle \in \mathcal{H}, \tag{2}$$

is *unitary.*

(Hint: First show that $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle$ for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$.)

b) Analogously show that an *antilinear*, invertible operator U, where antilinearity means

$$U\big(a|\psi\rangle + b|\phi\rangle\big) = a^*U|\psi\rangle + b^*U|\phi\rangle, \qquad |\psi\rangle, |\phi\rangle \in \mathcal{H}, \quad a, b \in \mathbb{C}, \tag{3}$$

is *antiunitary* (i.e. $UU^\dagger = U^\dagger U = \mathbf{1}$) if Eq. (2) is fulfilled. Note that for an antiunitary operator the definition of the adjoint operator is changed to $\langle U^\dagger\psi|\phi\rangle = \langle\psi|U\phi\rangle^*$.

Comment:
Wigner more generally showed that the requirement $|\langle U\psi|U\phi\rangle| = |\langle\psi|\phi\rangle|$ for all $|\psi\rangle, |\phi\rangle$ implies unitarity or antiunitarity of U (including linearity or antilinearity). A detailed, complete proof can be found in: S. Weinberg, *The Quantum Theory of Fields*, Vol. I, p. 91.

Please turn over!

Exercise 2.3 *Particle in a homogenous electric field* (4 points)

Consider a particle of mass m and electric charge q in a homogenous electric field of field strength ε in x direction. You need not consider the motion in the y and z directions in the following.

a) Formulate the time-independent Schrödinger equation in momentum representation for a fixed energy E. Determine the wave function $\langle k|E\rangle = \tilde{\psi}_E(k)$ of the energy eigenstate $|E\rangle$, where k is the usual wave number and $\tilde{\psi}_E(k)$ is related to the wave function $\langle x|E\rangle = \psi_E(x)$ in position space as follows,

$$\tilde{\psi}_E(k) \;=\; \int_{-\infty}^{+\infty} \mathrm{d}x \, \langle k|x\rangle \, \langle x|E\rangle \;=\; \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}kx} \, \psi_E(x). \tag{4}$$

Which E values are allowed?

b) Calculate the wave function $\psi_E(x)$ in position space from $\tilde{\psi}_E(k)$ upon inverting Eq. (4) and express the result in terms of the Airy function

$$\mathrm{Ai}(\xi) \;=\; \frac{1}{\pi} \int_0^{\infty} \mathrm{d}u \, \cos\left(\frac{u^3}{3} + \xi u\right) \, , \qquad \xi \in \mathbf{R}. \tag{5}$$

Normalize the wave functions according to

$$\langle E'|E\rangle \;=\; \int_{-\infty}^{+\infty} \mathrm{d}x \, \psi_{E'}(x)^* \psi_E(x) \;=\; \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \, \tilde{\psi}_{E'}(k)^* \tilde{\psi}_E(k) \;=\; \delta(E - E'). \tag{6}$$

c) Sketch $\psi_E(x)$ for a positive charge q graphically, making use of the asymptotics

$$\mathrm{Ai}(\xi) \underset{\xi \to +\infty}{\widetilde{\quad}} \frac{\mathrm{e}^{-\frac{2}{3}\xi^{3/2}}}{2\sqrt{\pi}\,\xi^{1/4}}, \qquad \mathrm{Ai}(\xi) \underset{\xi \to -\infty}{\widetilde{\quad}} \frac{\sin(\frac{2}{3}(-\xi)^{3/2} + \frac{1}{4}\pi)}{\sqrt{\pi}\,(-\xi)^{1/4}}, \tag{7}$$

respecting the signs of the $\psi_E''(x)/\psi_E(x)$ in the classically allowed / forbidden regions.

d) For the time $t = 0$ the particle is described by the wave function

$$\psi(x, 0) \;=\; \left(2\pi\sigma^2\right)^{-1/4} \exp\left\{-\frac{x^2}{4\sigma^2} + \mathrm{i}k_0 x\right\} \tag{8}$$

with some real constants σ and k_0. Calculate the expectation values $\langle\hat{x}\rangle_\psi$ and $\langle\hat{p}\rangle_\psi$ for any time t. What is the meaning of σ and k_0 ?

(Hint: Ehrenfest's theorem may save you from a lengthy calculation.)