Exercise 11.1 *Free-particle Green's function and propagator* (2 points + 1 bonus)

Green's functions for the time-independent Schrödinger equation are defined by

$$G^{\pm}(E,\vec{x},\vec{x}') \;=\; \langle\vec{x}|\,(E-\hat{H}\pm \mathrm{i}0)^{-1}\,|\vec{x}'\rangle,$$

where $\hat{H}$ is the (time-independent) Hamilton operator of the system. From $G^{\pm}(E,\vec{x},\vec{x}')$, Green's functions for the forward/backward evolution in time, the so-called retarded/advanced "propagators", are obtained as

$$G^{\pm}(\vec{x},t;\vec{x}',t') \;=\; \mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}\,\mathrm{e}^{-\mathrm{i}E(t-t')/\hbar}\,G^{\pm}(E,\vec{x},\vec{x}').$$

For the motion of a free particle (mass M) in three dimensions, calculate $G_0^{\pm}(\vec{x},t;\vec{x}',t')$ from

$$\begin{aligned} G_0^{\pm}(E,\vec{x},\vec{x}') &= \frac{\mathrm{i}}{(2\pi)^2|\vec{x}-\vec{x}'|}\int_{-\infty}^{\infty}\mathrm{d}k\,\frac{k\mathrm{e}^{-\mathrm{i}k|\vec{x}-\vec{x}'|}}{E-\frac{\hbar^2k^2}{2M}\pm\mathrm{i}0} \\ &= -\frac{M\mathrm{e}^{\pm\mathrm{i}k_E|\vec{x}-\vec{x}'|}}{2\pi\hbar^2|\vec{x}-\vec{x}'|}, \qquad k_E=\sqrt{2M(E\pm\mathrm{i}0)}/\hbar, \end{aligned}$$

which was derived in the lecture.

Hint: Do the integration over E first, so that the integration over k can be done with $\int_{-\infty}^{\infty}\exp\{-a(x+b)^2\}\,\mathrm{d}x=\sqrt{\pi/a}$ for $a,b\in\mathbf{C}$ with $\mathrm{Re}(a)>0$.
For the derivation of this auxiliary integral (for *complex* parameters a,b!) you may earn a bonus point.

Please turn over !

Exercise 11.2 *Spread of free wave packets* (3 points)

Consider the one-dimensional propagation of a free wave packet of mass m which is described by any normalized wave function $\psi(x,t)$.

a) Show that the momentum expectation value $\langle\hat{p}\rangle$ and momentum uncertainty $\Delta p \equiv \sqrt{\langle(\hat{p}-\langle\hat{p}\rangle)^2\rangle}$ are constant in time. How does the position expectation value $\langle\hat{x}\rangle$ develop in t?

b) Prove that the uncertainties Δx and Δp of position and momentum are related by

$$\Delta x^2 = \frac{\Delta p^2 t^2}{m^2} + at + \Delta x_0^2,$$

where Δx_0 is the spread at $t=0$ and a is a constant. Interpret the leading term for large times t.

c) Derive a bound on $|a|$ from Heisenberg's uncertainty principle. Which values can be taken by a if Δx_0 is minimal?

Exercise 11.3 *Free-particle wave functions with quantum numbers l, m* (3 points)

We consider the separation of the time-independent Schrödinger equation for a free particle of mass M in polar coordinates with the ansatz $\phi_{klm}(r,\theta,\varphi) = R_l(kr)Y_{lm}(\theta,\varphi)$ for the wave function. This leads to the differential equation

$$D^{(l)}R_l(\rho) \equiv \left[\frac{1}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\rho^2\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{l(l+1)}{\rho^2} + 1\right]R_l(\rho) = 0 \tag{1}$$

for the radial function $R_l(\rho) = R_l(kr)$, where $k \geq 0$ is related to the energy eigenvalue by $E(k) = \hbar^2k^2/(2M)$. As an ordinary 2nd-order differential equation, Eq. (1) possesses two linearly independent solutions for each value of $l = 0, 1, 2, \ldots$.

a) Show that the two independent solutions of Eq. (1) are given by

$$j_l(\rho) = (-\rho)^l\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\right)^l j_0(\rho), \qquad j_0(\rho) = \frac{\sin\rho}{\rho},$$

$$n_l(\rho) = (-\rho)^l\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\right)^l n_0(\rho), \qquad n_0(\rho) = -\frac{\cos\rho}{\rho}, \qquad l = 0, 1, \ldots,$$

where j_l and n_l are the spherical Bessel and Neumann functions, respectively.
Hint: A simple way is based on induction using $R_{l+1}(\rho) = -\rho^l\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho^{-l}R_l(\rho)\right]$ and evaluating the commutator of the differential operator $D^{(l+1)}$, as defined in Eq. (1), and the operator $\rho^l\frac{\mathrm{d}}{\mathrm{d}\rho}\rho^{-l}$.

b) Derive series expansions for j_l and n_l about $\rho = 0$, making use of the respective series for $\sin\rho$ and $\cos\rho$. Give the leading asymptotic behaviour of j_l and n_l for $\rho \to 0$.

c) Show that the leading asymptotic behaviour of j_l and n_l for $\rho \to \infty$ is given by

$$j_l(\rho) \underset{\rho\to\infty}{\sim} \frac{1}{\rho}\sin\left(\rho - \frac{l\pi}{2}\right), \qquad n_l(\rho) \underset{\rho\to\infty}{\sim} -\frac{1}{\rho}\cos\left(\rho - \frac{l\pi}{2}\right).$$