Exercise 7.1 *2-particle phase space* (1 point)

We consider two particles with masses $m_{1,2}$ and four-momenta $p_{1,2}$ ($p_{1,2}^2 = m_{1,2}^2$). The total momentum is, thus, given by $k = p_1 + p_2$. The integral over the 2-particle phase space is defined as

$$\int \mathrm{d}\Phi_2 \;=\; \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2p_1^0} \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2p_2^0} \,(2\pi)^4 \delta^{(4)}(k - p_1 - p_2) \bigg|_{p_{1,2}^0 = \sqrt{m_{1,2}^2 + \mathbf{p}_{1,2}^2}} .$$

a) Consider the decay of a particle of mass M and momentum k ($k^2 = M^2$) into two particles with momenta $p_{1,2}$. Show that the phase-space integral can be evaluated in the centre-of-mass frame as follows,

$$\int \mathrm{d}\Phi_2 \;=\; \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{8M^2} \,\theta(M - m_1 - m_2) \int \mathrm{d}\Omega_1,$$

where Ω_1 is the solid angle of particle 1, and $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$.

b) What is the phase-space integral of a $2 \to 2$ particle scattering reaction with incoming momenta $k_{1,2}$. What is the counterpart of M in this reaction?

Please turn over !

Exercise 7.2 *S-operator for two interacting scalar fields* (2 points)

Consider a theory of a complex scalar field ϕ (particle ϕ and antiparticle $\bar{\phi}$) and a real scalar field Φ (particle Φ) with the Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2 + (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2\phi^\dagger\phi + \lambda\phi^\dagger\phi\,\Phi\,.$$

a) Expand the S-operator,

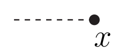
$$S = T\exp\left(\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}(x)\right),$$

up to order λ^2 and use Wick's theorem to express the result in terms of propagators and normal-ordered products of fields. Note that inside $\mathcal{L}_{\text{int}}$ all products of field operators are normal ordered. Represent the result diagrammatically using the following notation:

- External lines:

$\phi^\dagger(x) =$ 

, $\phi(x) =$ 

, $\Phi(x) =$ 

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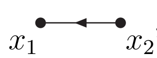
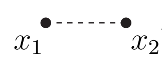
- Internal lines:

$\overline{\phi(x_1)\phi^\dagger}(x_2) =$ 

, $\overline{\Phi(x_1)\Phi}(x_2) =$ 

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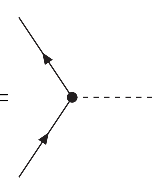
- Vertices:

$\mathrm{i}\lambda =$ 

The λ^n-contribution to the S-matrix is obtained by inserting the sum of all terms with n vertices into the integral

$$\frac{1}{n!}\int \mathrm{d}^4x_1\ldots d^4x_n : \ \ldots \ :$$

b) Calculate the S-matrix element $S_{fi} = \langle f|S|i\rangle$ in lowest order between the initial state $|i\rangle = a^{(\Phi)}(k)^\dagger|0\rangle$ and the final state $|f\rangle = a^{(\phi)}(p_1)^\dagger b^{(\phi)}(p_2)^\dagger|0\rangle$, where $a^{(\Phi)}(q)$, $a^{(\phi)}(q)$, $b^{(\phi)}(q)$ are the annihilation operators of the particles Φ, ϕ, and $\bar{\phi}$.

c) Assuming $M > 2m$, calculate the lowest-order decay width

$$\Gamma_{\Phi\to\phi\bar{\phi}} = \frac{1}{2M}\int \mathrm{d}\Phi_2\,|\mathcal{M}_{fi}|^2$$

for the decay $\Phi \to \phi\bar{\phi}$, where Φ_2 is the 2-particle phase space of the final state (see Exercise 7.1) and the transition matrix element $\mathcal{M}_{fi}$ is connected to S_{fi} via

$$S_{fi} = (2\pi)^4\delta(k - p_1 - p_2)\,\mathrm{i}\,\mathcal{M}_{fi}.$$