Exercises to Relativistic Quantum Field Theory **Sheet 10**

— Prof. S. Dittmaier, Dr. H. Rzehak, Universität Freiburg, SS14 —

Exercise 10.1 *Solutions of the free Dirac equation* (1 point)

Consider the solutions of the free Dirac equation in momentum space, $u_\sigma(k)$ and $v_\sigma(k)$ with $\sigma = 1, 2$, which are normalized according to

$$\overline{u}_\sigma(k)u_{\sigma'}(k) = -\overline{v}_\sigma(k)v_{\sigma'}(k) = 2m\,\delta_{\sigma\sigma'}, \qquad \overline{u}_\sigma(k)v_{\sigma'}(k) = \overline{v}_\sigma(k)u_{\sigma'}(k) = 0.$$

a) Prove the completeness relation

$$\sum_{\sigma=1,2} \left[u_\sigma(k) \otimes \overline{u}_\sigma(k) - v_\sigma(k) \otimes \overline{v}_\sigma(k)\right] = 2m\,\mathbf{1}.$$

b) Express the matrices

$$\Lambda_+(k) = \frac{1}{2m} \sum_{\sigma=1,2} u_\sigma(k) \otimes \overline{u}_\sigma(k), \qquad \Lambda_-(k) = -\frac{1}{2m} \sum_{\sigma=1,2} v_\sigma(k) \otimes \overline{v}_\sigma(k)$$

in terms of m, $\not{k}$ and the unit matrix $\mathbf{1}$. Argue that $\Lambda_\pm(k)$ are orthogonal projectors onto the subspaces of positive and negative energies, respectively.

c) Show that

$$\overline{u}_\sigma(k)\gamma_\mu u_{\sigma'}(k) = \overline{v}_\sigma(k)\gamma_\mu v_{\sigma'}(k) = 2k_\mu\,\delta_{\sigma\sigma'}$$

upon evaluating $\overline{u}_\sigma(k)\{\gamma_\mu, \not{k}\}u_{\sigma'}(k)$ in two different ways.

d) Similarly show that

$$v_\sigma(\tilde{k})^\dagger u_{\sigma'}(k) = u_\sigma(\tilde{k})^\dagger v_{\sigma'}(k) = 0$$

upon using $\gamma_0\not{k} = \tilde{\not{k}}\gamma_0$, where $\tilde{k}^\mu = (k_0, -\mathbf{k})$ for $k^\mu = (k_0, \mathbf{k})$.

Exercise 10.2 *Polarization sums for Dirac spinors* (0.5 points)

As in Exercise 10.1, the quantities $u_\sigma(p)$ and $v_\tau(P)$ with $\sigma, \tau = 1, 2$ denote the solutions of the free Dirac equation in momentum space, where $p^2 = m^2$, $P^2 = M^2$. Show that the following polarization sum $\sum_{\sigma,\tau}$ can be written as a trace in Dirac space according to

$$\sum_{\sigma,\tau} \Big[\bar{u}_\sigma(p)\Gamma v_\tau(P)\Big]^* \Big[\bar{u}_\sigma(p)\Gamma v_\tau(P)\Big] = \mathrm{Tr}\Big[(\not{P} - M)\tilde{\Gamma}(\not{p} + m)\Gamma\Big].$$

Here Γ is an arbitrary 4×4 matrix and $\tilde{\Gamma} = \gamma_0\Gamma^\dagger\gamma_0$.

Please turn over !

Exercise 10.3 *Field operator of the free Dirac fermion* (1 point)

Consider the following plane-wave expansion of the field operator $\psi(x)$ of the free Dirac fermion,

$$\psi(x) = \int \mathrm{d}\tilde{p} \sum_{\sigma} \left[\mathrm{e}^{-\mathrm{i}px} u_\sigma(p) a_\sigma(\mathbf{p}) + \mathrm{e}^{+\mathrm{i}px} v_\sigma(p) b_\sigma^\dagger(\mathbf{p}) \right],$$

where $a_\sigma^{(\dagger)}(\mathbf{p})$ and $b_\sigma^{(\dagger)}(\mathbf{p})$ denote the annihilation (creation) operators of the particle and antiparticle states, respectively, which obey the anticommutation relations

$$\{a_\sigma(\mathbf{p}), a_\tau^\dagger(\mathbf{k})\} = \{b_\sigma(\mathbf{p}), b_\tau^\dagger(\mathbf{k})\} = 2p_0(2\pi)^3 \delta_{\sigma\tau} \delta(\mathbf{p} - \mathbf{k}), \qquad \{a_\sigma(\mathbf{p}), a_\tau(\mathbf{k})\} = \cdots = 0.$$

a) Calculate the operator

$$\hat{Q} = Qe \int \mathrm{d}^3x \; : \overline{\psi}(x) \gamma_0 \psi(x) :$$

of electric charge in momentum space.

b) Derive the commutators $[\hat{Q}, a_\sigma^{(\dagger)}(\mathbf{p})]$ and $[\hat{Q}, b_\sigma^{(\dagger)}(\mathbf{p})]$ and interpret the result.

Exercise 10.4 *Free Dirac propagator* (1 bonus point)

The free Dirac propagator is explicitly given by

$$S_F(x,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \, \mathrm{e}^{-\mathrm{i}k(x-y)} \, \frac{\not{k} + m}{k^2 - m^2 + \mathrm{i}\epsilon}.$$

a) Upon carrying out the k_0-integration, show that $S_F(x,y)$ can be written as

$$\begin{aligned} S_F(x,y) \;\; &= \;\; -\mathrm{i}\theta(x_0 - y_0) \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \, \mathrm{e}^{-\mathrm{i}k(x-y)} (m + \not{k}) \Big|_{k_0 = \sqrt{m^2 + \mathbf{k}^2}} \\ &\quad\; -\mathrm{i}\theta(y_0 - x_0) \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \, \mathrm{e}^{+\mathrm{i}k(x-y)} (m - \not{k}) \Big|_{k_0 = \sqrt{m^2 + \mathbf{k}^2}}. \end{aligned}$$

b) We denote the solutions of the free Dirac equation with positive and negative energies $\psi^{(+)}(x)$ and $\psi^{(-)}(x)$, respectively. Show that

$$\begin{aligned} \theta(x_0 - y_0)\psi^{(+)}(x) &= \mathrm{i} \int \mathrm{d}^3y \, S_F(x,y) \gamma_0 \psi^{(+)}(y), \\ \theta(y_0 - x_0)\psi^{(-)}(x) &= -\mathrm{i} \int \mathrm{d}^3y \, S_F(x,y) \gamma_0 \psi^{(-)}(y). \end{aligned}$$