Exercises "Modern methods of Quantum Chromodynamics" **WS 14/15**

Problem 18 (4 Points) *Scalar-gluon scattering*

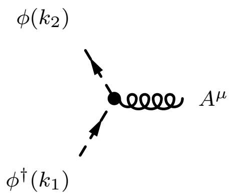
Consider a theory with a complex scalar field ϕ_i in the fundamental representation of $SU(3)$. The four-point born amplitudes with two scalars and two gluons admit the colour decomposition into colour-ordered partial amplitudes

$$\mathcal{M}(\phi_1^{i,\dagger}, \phi_{j,2}, g_{a,3}, g_{b,4}) = g_s^2 (T^b T^a)^i_j M(\phi_1^\dagger, \phi_2, g_3, g_4) + g_s^2 (T^a T^b)^i_j M(\phi_1^\dagger, \phi_2, g_3, g_4)$$

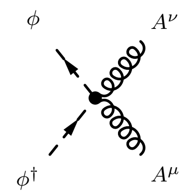
Compute the partial amplitudes

$$M(\phi_1^\dagger, \phi_2, g_3^+, g_4^+)\,, \qquad\qquad M(\phi_1^\dagger, \phi_2, g_3^+, g_4^-).$$

The colour-ordered Feynman rules with outgoing momenta are given by



$: -\mathrm{i}(k_1 - k_2)^\mu\,,$



$: \quad \mathrm{i}g^{\mu\nu}$

and the usual colour-ordered Feynman rules for QCD.

Problem 19 (2 Points) *Amplitude relations*

Colour-ordered gluon amplitudes satisfy the so-called *dual Ward identity*

$$M_n(g_1, g_2, g_3, \ldots g_n) + M_n(g_2, g_1, g_3, \ldots g_n) + M_n(g_2, g_3, g_1, \ldots g_n) + \cdots + M_n(g_2, g_3, g_1, \ldots g_1, g_n) = 0$$

Check that this identity is satisfied for the case of maximally helicity-violating amplitudes

$$M_n(g_1^+, \ldots, g_i^-, \ldots, g_j^-, \ldots g_n^+) = 2^{n/2-1} \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

Problem 19 (4 Points) *Colour factors*

Compute the following products of traces over $SU(N_c)$ generators

$$\begin{aligned}
\text{tr}\left[T^a T^b\right] \text{tr}\left[T^b T^a\right] &= \frac{N_c^2-1}{4} \\
\text{tr}\left[T^a T^b T^c\right] \text{tr}\left[T^c T^b T^a\right] &= \frac{(N_c^2-1)(N_c^2-2)}{8N_c} \\
\text{tr}\left[T^a T^b T^c\right] \text{tr}\left[T^a T^b T^c\right] &= -\frac{N_c^2-1}{4N_c} \\
\text{tr}\left[T^a T^b T^c T^d\right] \text{tr}\left[T^d T^c T^b T^a\right] &= \frac{N_c^6-4N_c^4+6N_c^2-3}{16N_c^2}
\end{aligned}$$

Discuss the behaviour for $N_c \to \infty$.

Bonus question (1 bonus point)
Compute also

$$\text{tr}\left[T^a T^b T^c T^d\right] \text{tr}\left[T^a T^b T^c T^d\right]$$

Show first that

$$\text{tr}[T^a T^b T^c T^a T^b T^c] = \frac{N_c^4-1}{8N_c^2}$$