Problem 12 (4 Points) *Spinor identities*

a) Derive the following identities:

$$[p|\gamma^\mu|k\rangle = \langle k|\gamma^\mu|p],$$
$$\langle p|\gamma^\mu\gamma^\nu|k\rangle = -\langle k|\gamma^\nu\gamma^\mu|p\rangle .$$

b) For a set of n light-like momenta $k_1, \ldots k_n$ and an arbitrary spinor $|q\rangle$, show the so-called eikonal identity:

$$\sum_{i=1}^{n-1} \frac{\langle k_i k_{i+1}\rangle}{\langle k_i q\rangle \langle q k_{i+1}\rangle} = \frac{\langle k_1 k_n\rangle}{\langle k_1 q\rangle \langle q k_n\rangle}.$$

Problem 13 (4 Points) *Polarization vectors*

The gluon polarization vectors can be defined in terms of reference spinors $|q\rangle$ and $[q|$ as

$$\epsilon_+^\mu(k,q) = \frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}\,[kq]}, \qquad \epsilon_-^\mu(k,q) = \frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\,\langle qk\rangle},$$

with $\epsilon_\pm^* = \epsilon_\mp$. Derive the following properties:

a) Normalization:

$$\epsilon_\lambda(k,q)\cdot\epsilon_{\lambda'}^*(k,q) = -\delta_{\lambda,\lambda'}.$$

b) Gauge transformation:

$$\epsilon_-^\mu(k,q) - \epsilon_-^\mu(k,q') = \sqrt{2}\frac{\langle q'q\rangle}{\langle q'k\rangle\langle qk\rangle}\, k^\mu.$$

c) Completeness relation

$$\sum_{\lambda=\pm} \epsilon_\lambda^\mu(k,q)\epsilon_\lambda^{\nu *}(k,q) = -g^{\mu\nu} + \frac{k^\mu q^\nu + q^\mu k^\nu}{(k\cdot q)}.$$

Problem 14 (2 Points) *Momentum conservation*

For a $2 \to 2$ scattering process of massless particles with incoming momenta $p_{1/2}$ and outgoing momenta $k_{1/2}$ with momentum conservation, $p_1+p_2 = k_1+k_2$, derive the identities

$$\langle p_1 k_1\rangle [k_1 p_2] = -\langle p_1 k_2\rangle [k_2 p_2],$$
$$\langle p_1 p_2\rangle [p_2 k_1] = \langle p_1 k_2\rangle [k_2 k_1].$$

Spinor formulas

- Notations for Weyl spinors:

$$p_A \leftrightarrow |p+\rangle = |p\rangle \qquad p^{\dot{A}} \leftrightarrow |p-\rangle = |p],$$
$$p_{\dot{A}} \leftrightarrow \langle p+| = [p| \qquad p^A \leftrightarrow \langle p-| = \langle p|$$

- Antisymmetric symbol

$$\varepsilon^{AB} = \varepsilon_{AB} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- Raising and lowering indices

$$p^A = \varepsilon^{AB} p_B, \qquad p_{\dot{B}} = p^{\dot{A}} \varepsilon_{\dot{A}\dot{B}}$$
$$p_B = p^A \varepsilon_{AB}\,, \qquad p^{\dot{A}} = \varepsilon^{\dot{A}\dot{B}} p_{\dot{B}},$$

- Spinor products:

$$\langle pk\rangle = \langle p-|k+\rangle = p^A k_A \qquad [kp] = \langle k+|p-\rangle = k_{\dot{A}} p^{\dot{A}}$$

- Properties of Pauli matrices:

$$\sigma^{\mu}_{\dot{A}B} = \sigma^{\mu,\dot{B}C} \varepsilon_{\dot{B}\dot{A}} \varepsilon_{CD} \qquad \bar{\sigma}^{\mu,A\dot{B}} = \varepsilon^{AC} \varepsilon^{\dot{B}\dot{D}} \bar{\sigma}_{\mu,C\dot{D}}$$
$$\sigma^{\mu}_{\dot{A}B} = (\bar{\sigma}^{\mu}_{A\dot{B}})^* = \bar{\sigma}^{\mu}_{B\dot{A}} \qquad \bar{\sigma}^{\mu,A\dot{B}} = (\sigma^{\mu,\dot{A}B})^* = \sigma^{\mu,\dot{B}A}$$

- Matrix elements of Pauli matrices:

$$p_{\dot{A}} \sigma^{\mu,\dot{A}B} k_B \equiv \langle p+|\gamma^\mu|k+\rangle = [p|\gamma^\mu\,|k\rangle\,,$$
$$p^A \bar{\sigma}^{\mu}_{A\dot{B}} k^{\dot{B}} \equiv \langle p-|\gamma^\mu|k-\rangle = \langle p|\,\gamma^\mu|k]$$

- Schouten identity: $\varepsilon^{AB}\varepsilon^{CD} + \varepsilon^{AC}\varepsilon^{DB} + \varepsilon^{AD}\varepsilon^{BC} = 0.$

  Relations for spinor products:

$$|p\rangle\,\langle kq\rangle + |k\rangle\,\langle qp\rangle + |q\rangle\,\langle pk\rangle = 0,$$
$$|p]\,[kq] + |k]\,[qp] + |q]\,[pk] = 0.$$

- Fierz identity:

$$\langle p|\,\gamma^\mu|k]\,\langle q|\,\gamma_\mu|l] = 2\,\langle pq\rangle\,[lk]\,, \qquad [p|\gamma^\mu\,|k\rangle\,[q|\gamma_\mu\,|l\rangle = 2\,[pq]\,\langle lk\rangle$$