Exercises "Modern methods of Quantum Chromodynamics" WS 14/15

Problem 1 (1 Point) *$SU(3)$ transformations*

The quantum states of quarks $|q\rangle$ and antiquarks $|\bar{q}\rangle$ carry a colour index $i = 1, 2, 3$ and transform under rotations in colour space as

$$|q_i\rangle \to U_i^{\ j}\,|q_j\rangle \qquad\qquad |\bar{q}^i\rangle \to |\bar{q}^j\rangle\,(U^\dagger)_j^{\ i}$$

with unitary 3-by-3 matrices U with unit determinant. Show that the state

$$|D^i\rangle \equiv \epsilon^{ijk}\,|q_j\rangle \otimes |q_k\rangle$$

transforms in the same way as an antiquark.

Problem 2 (4 Points) *Pauli Matrices and spinors*

The spinors of left- and right-handed massless, relativistic spin 1/2 particles are given by

$$u_R(p) = \begin{pmatrix} u_+(p) \\ 0 \end{pmatrix}, \qquad u_+ = \frac{1}{\sqrt{(p^0 - p^3)}}\begin{pmatrix} p^1 - \mathrm{i}p^2 \\ p^0 - p^3 \end{pmatrix},$$

$$u_L(p) = \begin{pmatrix} 0 \\ u_-(p) \end{pmatrix}, \qquad u_- = \frac{1}{\sqrt{(p^0 - p^3)}}\begin{pmatrix} p^0 - p^3 \\ -(p^1 + \mathrm{i}p^2) \end{pmatrix}.$$

a) Show that the above spinors are solutions to the massless Dirac equation in momentum space,

$$\gamma^\mu p_\mu\, u_\lambda(p) = 0,$$

for $p_\mu p^\mu = 0$. The gamma matrices and sigma matrices are defined as

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \qquad \sigma^\mu = (\mathbf{1}, \vec{\sigma}), \qquad \bar{\sigma}^\mu = (\mathbf{1}, -\vec{\sigma}).$$

b) Verify that the spinors satisfy the normalization condition

$$\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = 2p^\mu\,\delta_{\sigma\sigma'} \quad \text{with} \quad \sigma, \sigma' = L/R.$$

c) Derive the identity

$$\sigma^\mu_{ab}\bar{\sigma}_{\mu,cd} = 2\delta_{ad}\delta_{bc}$$

Here the indices $a = 1, 2$, etc. are spinor indices. You can use the normalization of the Pauli matrices, $\mathrm{tr}(\sigma^\mu\bar{\sigma}^\nu) = 2g^{\mu\nu}$, and the fact that any hermitian two-by-two matrix A can be written as a linear combination of Pauli matrices and the unit matrix, $A_{ab} = A_0\delta_{ab} + \sum_i A_i\sigma^i_{ab} \equiv A_\mu\sigma^\mu_{ab}$.

d) Derive the expression

$$2p_\mu k^\mu = \langle pk\rangle\,[kp]$$

for light-like momenta p, k, with the spinor products

$$\langle pk\rangle \equiv \bar{u}_L(p)u_R(k) = u_-^\dagger(p)u_+(k), \qquad [kp] \equiv \bar{u}_R(k)u_L(p) = u_+^\dagger(k)u_-(p).$$

Problem 3 (3 Points) *Pauli-Lubanski Vector*

Verify the commutation relations of the Pauli-Lubanski vector $W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ with the generators of the Poincaré algebra:

$$[W^\mu, P^\nu] = 0, \qquad [W^\mu, M^{\rho\sigma}] = i\left(g^{\mu\rho}W^\sigma - g^{\mu\sigma}W^\rho\right).$$

You can use the Poincaré Algebra:

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= -i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}\right), \\
[P^\mu, P^\nu] &= 0, \\
[P^\mu, M^{\rho\sigma}] &= i\left(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho\right).
\end{aligned}$$

and the identity

$$0 = g_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma} + g_{\alpha\mu}\epsilon_{\nu\rho\sigma\beta} + g_{\alpha\nu}\epsilon_{\rho\sigma\beta\mu} + g_{\alpha\rho}\epsilon_{\sigma\beta\mu\nu} + g_{\alpha\sigma}\epsilon_{\beta\mu\nu\rho}$$

Problem 4 (2 Points) *Vector-boson fields in axial Gauge*

Consider free photon Lagrangian in QED with an *axial gauge* fixing-term:

$$\begin{aligned}
\mathcal{L}_A &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{\text{fix}}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\
\mathcal{L}_{\text{fix}} &= -\frac{1}{2\xi}(n^\mu A_\mu)^2,
\end{aligned}$$

where n^μ is a constant four-vector. Compute the free photon propagator in momentum space $D_F^{\mu\nu}(p)$ for this gauge.

Hints: Writing the free photon Lagrangian for the field A^μ in the form

$$\mathcal{L}_A = \frac{1}{2}A_\mu(x)\mathcal{D}_x^{\mu\nu}(x)A_\nu(x),$$

the propagator in position space is the Green function for the operator $\mathcal{D}_{\mu\nu}$:

$$\mathcal{D}_x^{\mu\nu}D_{F,\nu\rho}(x,y) = \mathrm{i}g^\mu_\rho\delta^4(x-y).$$

The propagator in momentum space is given by a Fourier transform:

$$\mathrm{i}D_F^{\mu\nu}(x,y) = \int\frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}D_F^{\mu\nu}(p).$$

Make an Ansatz for the tensor structure of $D_F^{\mu\nu}(p)$.