Modern Methods of Quantum Chromodynamics

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Contents

Chapter 1

Introduction

Hadrons and quarks

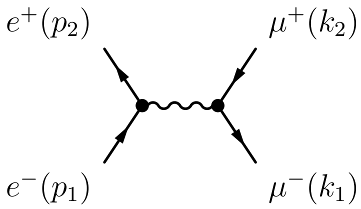
In the 1960s and early 1970s it was discovered that Hadrons (protons, neutrons, pions. . .) are **composite** particles, composed of quarks: spin 1/2 particles with non-integer electric charges. In Chapter 2 we will briefly review the evidence for this picture.

QFT and QED

A consistent description of relativistic quantum processes is given by **Quantum field theory**: the Hamiltonian of the theory is constructed in terms of field operators $\Phi(x)$, which create or destroy particles/antiparticles at the space-time point $x = (t, \vec{x})$.

The relativistic QFT describing electromagnetic interactions of electrons, positrons and other charged particles is called **Quantum Electrodynamics** (QED). It combined the quantized Maxwell theory with the **Dirac equation** describing relativistic spin 1/2-particles. The excitations of the quantized electromagnetic field are called **photons**.

The description of scattering process in QFT in perturbation theory can be described in terms of **Feynman diagrams**. The **Feynman rules** give a precise description to translate the diagram



to a mathematical expression for the scattering $e^-e^+ \to \mu^- \to \mu^-$. The basics of QFT, QED and Feynman diagrams are reviewed in Section 3.2.

QCD: theory of quarks and gluons

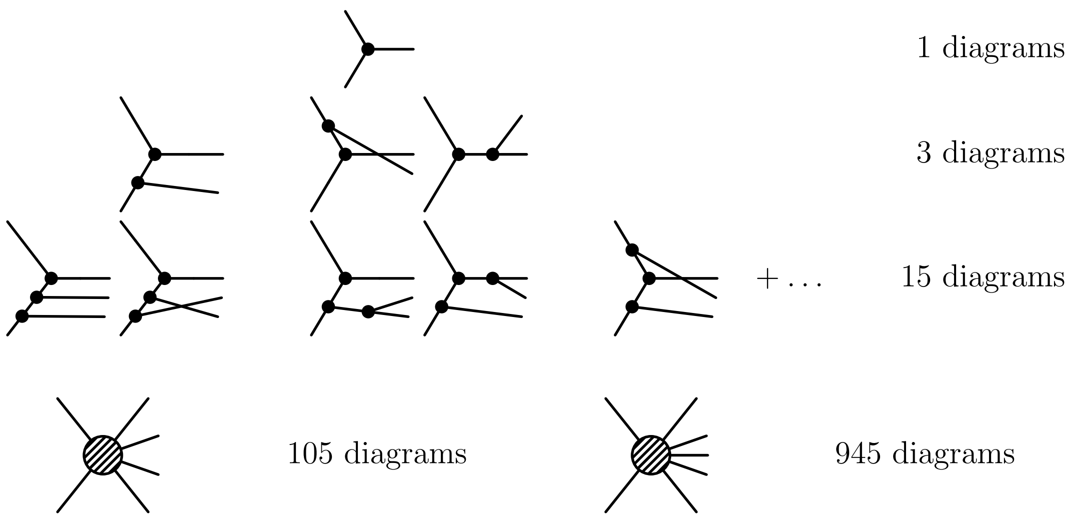
Quantum Chromodynamics (QCD) is the theory describing the strong interactions among quarks. Its mathematical structure is an extension of QED. Similar to QED, where the electrodynamic interactions are described by an exchange of photons, in QCD the strong interactions of quarks is described by the exchange of gluons, massless spin-1 particles, as the photons. As QED, QCD is a **gauge theory**, i.e. the Lagrangian is invariant under gauge transformations. The analog of the electric charge is a quantum number called **colour**. Whereas photons are not electrically charged, gluons also carry colour quantum numbers, so they are self-interacting. The basics of QCD and the Feynman rules are discussed in Chapter 4

QCD and LHC physics

The theoretical description of any scattering process at a hadron-hadron collider, such as the Tevatron running until 2012 at Fermilab near Chicago or the Large Hadron running at CERN in Geneva since 2010, requires to calculate cross sections in QCD. Typical processes involve the production of **jets**, collimated collections of hadrons arising from the hadronization of quarks or gluons. In the quark-parton model, one important ingredient for the prediction of jet production are the "partonic cross sections" for quark and gluon production. Examples for the simplest case of dijet-production will be given in Chapter **??**.

Multi-parton scattering amplitudes

At LHC, cross sections for processes with a large number of jets can be measured, see Figure 1.1. In principle, the calculation of scattering amplitudes for such processes using textbooks methods is possible, but made impractical by the rapidly growing number of Feynman diagrams:

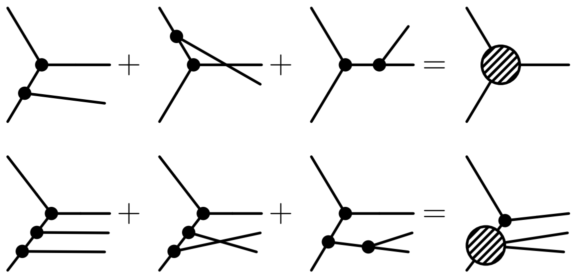


Since the mid-1980s, methods for the calculations for such amplitudes have been developed. These will be the focus of Part II.

- The use of spinor methods to compute amplitudes with a fixed assignment of helicities of the external particles. For a class of amplitudes involving an arbitrary number of gluons (so-called “maximally helicity violating” amplitudes), a simple-one-line formula was conjectured by Parke/Taylor (1986)

$$|A(g_1^+ \dots g_i^-, \dots g_j^- \dots g_n^+)|^2 \sim \frac{(p_i \cdot p_j)^4}{(p_1 \cdot p_2)(p_2 \cdot p_3) \dots (p_n \cdot p_1)} \tag{1.1}$$

- The Parke-Taylor formula was proven by a recursive construction (Berends/Giele 1987) where sub-diagrams appearing in many diagrams are computed only once:



  Such a recursive construction can be efficiently implemented in a computer program and is used for most of the theoretical predictions compared to the experimental results in figure 1.1.

- More activity in the field of multi-parton scattering amplitudes was triggered in 2003 by a paper by Witten exploring the formal structure of scattering amplitudes and relations to twistor and string theory. This led to several new alternative constructions of scattering amplitudes: one method using the MHV amplitudes (1.1) as building blocks, and one method constructing scattering amplitudes recursively from on-shell scattering amplitudes with fewer legs (in contrast to Feynman diagrams or the Berends/Giele relations where the internal legs are off-shell).

NLO calculations

For reliable predictions of cross sections, at least the next-to-leading order in perturbation theory in QCD has to be computed, corresponding to Feynman diagrams with a closed loop. This requires methods for the computation of the resulting integrals, including regularization of divergences in intermediate steps and renormalization to absorb the divergences in relations among observable quantities. The **renormalization group** allows to relate observables at different scales. In recent years alternatives to Feynman diagrams have been developed, so called **unitarity methods** that allow to compute Loop amplitudes in terms of tree-level amplitudes. These topics are discussed in Part III.

Remarks on the lecture

Part I and the first chapters of Part III contain topics that are usually discussed in a lecture on QCD and collider physics, and also contain a brief review of QFT, QED and Feynman

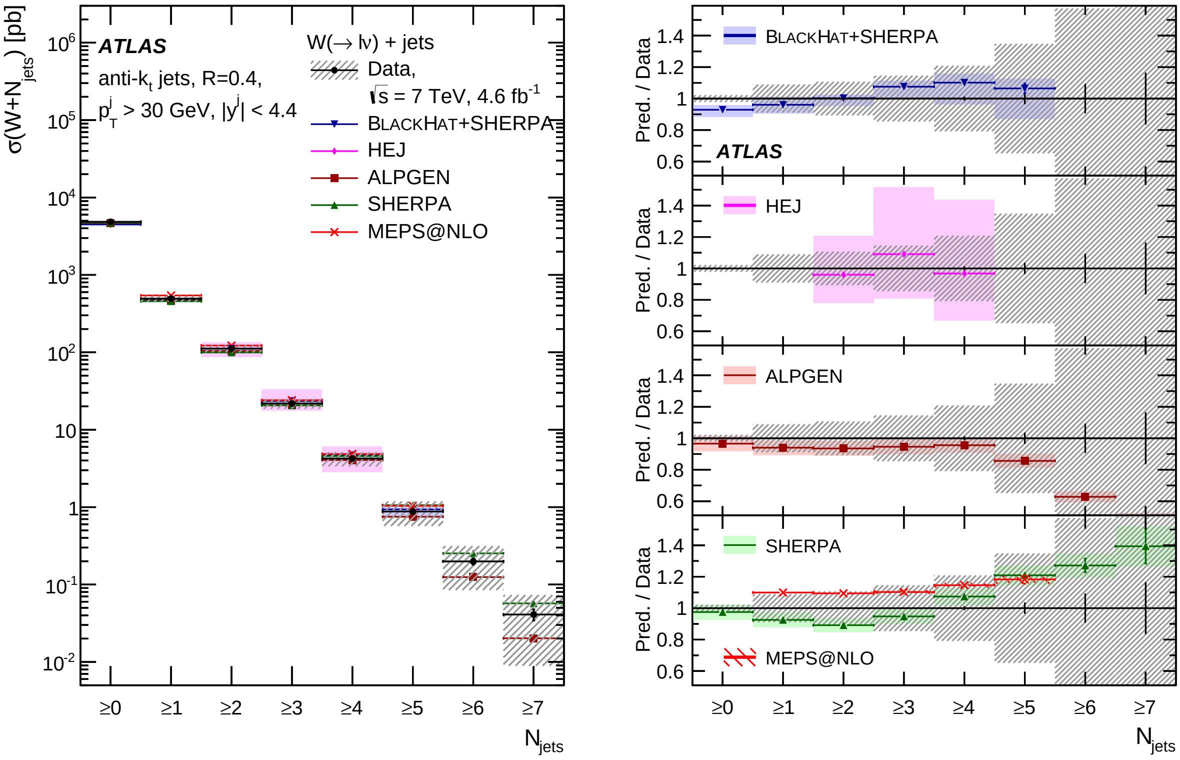


Figure 1.1: Experimental results for the production of a W boson in association with N_{jets} jets from the ATLAS experiment at the LHC. Taken from `http://arxiv.org/abs/arXiv:1409.8639`

diagrams. The methods introduced in Part II are usually not discussed in introductory lectures, but have become an important ingredient of research in QCD in the last 10 years. A very recent textbook that covers most of the contents of the lecture is [1], which also contains a didactic introduction to the basics of Quantum Field Theory. The conventions in Quantum field theory used here mainly follow [2]. An interesting introduction to QCD can be found online in the lecture notes [6] and a useful book on the basics and applications of QCD is [7]. Lecture notes on the more recent developments in QCD are also available online [8, 9]. More advanced extensive reviews are also available focusing on NLO calculations [10] or applications to supersymmetric theories [11] and also contain useful material for the lecture. One part of a usual lecture on QCD that cannot be covered here is the proper quantization, which requires to introduce the path- (functional) integral method. Here the results are quoted in Chapter 4.

Part I

Parton Model and QCD

Chapter 2

Quarks and colour

In this chapter we give a brief overview over the evidence for quarks and the colour quantum number. See e.g. [7] for a similar discussion.

2.1 Hadrons and quarks

Hadrons and the strong interactions

- In an attempt to model the **strong interactions** of protons and neutrons in analogy to QED, the spin zero charged and neutral **pions** were proposed as messenger particles by Yukawa in 1935, and experimentally discovered in 1947. Since the strong interaction is short-ranged, the pions are massive ($m_{\pi_0} = 135$ MeV, $m_{\pi^\pm} = 139$ MeV. However, a successful QFT for the strong interactions could not be constructed.

- Subsequently more **mesons** (integer spin) and **baryons** (half-integer spin) were discovered, e.g. Kaons:($K^\pm$, K^0, $\bar{K}^0$, $m_K \sim 490$ MeV, discovery 1947), rho mesons (ρ^0, $\rho^\pm$, spin 1, $m_\rho = 770$ MeV, discovery 1961), Δ baryons (Δ^{++}, Δ^+, Δ^0, Δ^-, spin 3/2, mass $m_\Delta \sim 1.2$ GeV.

$\Rightarrow$ "Particle zoo"

- Classification by "isospin" and "strangeness" (I, S) (e.g. p: $(+\frac{1}{2}, 0)$, $n = (-\frac{1}{2}, 0)$, $\pi^{\pm,0}$: $(\pm 1/0, 0)$, $K^\pm$: $(\pm\frac{1}{2}, \mp 1)$, K^0: $(-\frac{1}{2}, -1)$, $\bar{K}^0$: $(\frac{1}{2}, 1)$)

Quark Model

- Gell-Mann, Ne'eman, Zweig (1961-64) classified the hadron spectrum and proposed that hadrons are composed of spin 1/2 **quarks** with fractional electric charges:

$$u(I = +\tfrac{1}{2}, S = 0, Q = \tfrac{2}{3}), \qquad d(-\tfrac{1}{2}, 0, -\tfrac{1}{3}), \qquad s(0, -1 - \tfrac{1}{3}), \tag{2.1}$$

- In the quark picture, baryons (proton, neutron, Δ,...) are composed of three quarks:

$$|p\rangle \sim |uud\rangle \qquad\qquad |n\rangle \sim |udd\rangle \tag{2.2}$$

- Mesons (pions, kaons,...) are composed of quark-antiquark pairs:

$$|\pi^+\rangle \sim |u\bar{d}\rangle \qquad |\pi^-\rangle \sim |d\bar{u}\rangle \qquad |\pi^0\rangle \sim |d\bar{d}\rangle - |u\bar{u}\rangle\,, \tag{2.3}$$

$$|K^+\rangle \sim |u\bar{s}\rangle \qquad |K^-\rangle \sim |s\bar{u}\rangle \qquad |K^0\rangle \sim |d\bar{s}\rangle \qquad |\bar{K}^0\rangle \sim |s\bar{d}\rangle \tag{2.4}$$

- The observed hadron masses can be explained by assuming u, d, s are approximately mass-degenerate, treating the mass splitting $m_s - m_{d/u}$ as a perturbation. This allows the classification of hadrons into approximate $SU(3)$ multiplets.[1]

- The quark model allowed the prediction of "missing" states such as the $|\Omega^-\rangle \sim |sss\rangle$ spin 3/2 baryon, subsequently discovered 1964.

- no free quarks are observed in nature $\Rightarrow$ **confinement** hypothesis: interaction among quarks is so strong, that quarks are always bound together.

- Subsequently: discovery of charm, bottom and top quarks:

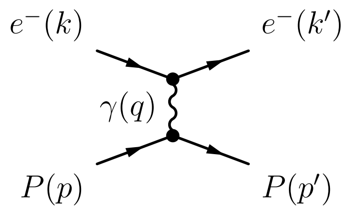
$$\begin{array}{rcr} & Q, & m \\ d: & -\frac{1}{3}, & 5\text{MeV} \\ u: & +\frac{2}{3}, & 2.5\text{MeV} \\ s: & -\frac{1}{3}, & 100\text{MeV} \\ c: & +\frac{2}{3}, & 1.3\text{GeV} \\ b: & -\frac{1}{3}, & 4.7\text{GeV} \\ t: & +\frac{2}{3}, & 173.3\text{GeV} \end{array} \tag{2.5}$$

All quarks apart from the u and d quarks are unstable and decay to lighter quarks.

2.2 Parton Model

Deep inelastic scattering

Electron proton scattering:



[1] Historically, this classification was performed before the quarks were postulated. The "flavour $SU(3)$" of the hadron spectrum should not be confused with the colour-$SU(3)$ discussed below.

Usually parametrized by **momentum transfer** Q^2 and **energy transfer** ν

$$Q^2 = -q^2 \equiv -(k-k')^2 \qquad\qquad \nu = k^0 - k'^0 \equiv E - E' \tag{2.6}$$

In the restframe of the initial-state proton, the proton four-momentum is

$$p^\mu = \big(m_P, 0, 0, 0\big) \tag{2.7}$$

Energy conservation determines the energy of the final-state proton as

$$p'^0 = m_P + E - E', \tag{2.8}$$

Using four-momentum conservation gives for the momentum transfer

$$Q^2 = -(p-p')^2 = 2p\cdot p' - m_P^2 - p'^2 = 2m_P\nu + (m_P^2 - p'^2) \tag{2.9}$$

For elastic electron-proton scattering $p'^2 = m_P^2$ so that the **Bjorken** variable

$$x = \frac{Q^2}{p\cdot q} = \frac{Q^2}{2m_P\nu} \tag{2.10}$$

is equal to 1.

Experimentally, a different behaviour of the scattering cross section is observed depending on momentum transfer:

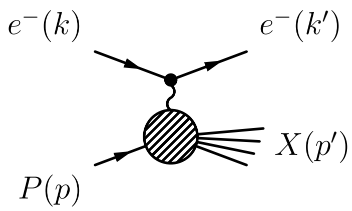
Elastic scattering: $Q^2 \lesssim .01\,\mathrm{GeV}^2$**:** The scattering cross section is modified compared to that of the scattering of an electron off a point-like spin one-half particle. This can be described in terms of **formfactor** depending on Q^2/Λ^2 with $\Lambda \lesssim 1\,\mathrm{GeV}$. The formfactor is related to the Fourier-transform of the charge distribution. Therefore the momentum scale Λ implies a finite proton radius $\sim 1\,\mathrm{fm}$.

Inelastic scattering: $Q^2 \gtrsim 0.1\,\mathrm{GeV}^2$ The energy is high enough to produce new particles, e.g. $e^-P \to e^-P\pi^0$, $e^-P \to e^-\Delta^+ \to e^-N\pi^+$,

Deep inelastic scattering: $Q^2 > 1\,\mathrm{GeV}^2$ For very high energies the proton is disintegrated completely by the scattering and a large number of hadrons is produced. In this case we are interested in the "inclusive" cross section for the process

$$e^-p \to e^-X$$

where X denotes the complete hadronic final state.



In this case $p'^2 = m_X^2 \neq m_P^2$ so that the Bjorken variable satisfies

$$0 < x = \frac{Q^2}{2m_P\nu} = 1 + \frac{m_P^2 - m_X^2}{2m_P\nu} \leq 1 \tag{2.11}$$

The ratio x can be measured just from knowledge of the electron energies and the scattering angle $\theta = \angle(\vec{p}, \vec{p}')$ since

$$Q^2 = 2k \cdot k' = 2EE'(1 - \cos\theta^2) = 4EE' \sin^2\frac{\theta}{2} \tag{2.12}$$

- The observed cross sections for DIS showed no prominent resonances as for inelastic scattering at smaller Q^2 but a continuum similar to scattering from a point-like particle.
- This was interpreted by Bjorken/Paschos (1969) and Feynman (1972) assuming the proton consists of light, quasi-free particles, called **partons**. Subsequently the partons were identified with the quarks and gluons.

Parton distribution functions

In a reference frame where the proton moves at a high energy so that the proton mass can be neglected (the so-called **infinite momentum frame**), the proton momentum can be approximated as $p^\mu = p(1, 0, 0, 1)$. In this frame the parton i carries momentum

$$p_i^\mu = \xi_i p^\mu \tag{2.13}$$

One introduces the probability to find a quark with momentum fraction ξ_i in the interval $\xi_i + d\xi_i$ as

$$f_i(\xi_i)\mathrm{d}x_i \tag{2.14}$$

where $f_i(\xi x)$ is the so-called **parton-distribution function** for parton i in the proton. The on-shell condition of the scattered quark,

$$p_i^2 = (p_i + q)^2 \tag{2.15}$$

implies

$$2p_i \cdot q = 2\xi_i\, p \cdot q = -q^2 = Q^2 \tag{2.16}$$

This determines the momentum fraction of the parton in the proton in terms of the Bjorken variable:

$$\xi_i = \frac{Q^2}{2(p \cdot q)} = x \tag{2.17}$$

The cross section for DIS is written in the parton model as a incoherent sum over "partonic cross sections"

$$\sigma(e^- p \to e^- X) = \int_0^1 \mathrm{d}x \sum_i f_i(x)\, \hat{\sigma}(e^- q_i \to e^- q_i) \tag{2.18}$$

where the sum is over the quark flavours i. At leading order, only the quarks are relevant for DIS, since the gluons are not electrically charged.

- In the naive parton model, the PDFs depend only on x, and on no dimensionful quantity (for instance not on the proton radius). This property was observed to be approximately satisfied in the first DIS experiments.

- In the full QCD treatment, the PDFs depend also on the scale Q^2 of the process, in agreement with experiments.

- The precise form of the cross section is also a test of the spin 1/2 nature of the quarks, since the partonic cross section

DIS in the parton model is discussed in more detail in Chapter 5

2.3 Colour degree of freedom

Postulate of colour quantum number

The naive quark model has several problems, for example:

- it is not explained why only qqq and $q\bar{q}$ states are observed .

- it led to difficulty describing the quantum numbers of some particles.

An example for the latter problem is given by baryons with three identical quarks, e.g. the Δ^{++}:

$$|\Delta^{++}\rangle \sim |uuu\rangle \tag{2.19}$$

- The Pauli principle implies that wave function of identical fermions must be antisymmetric under the exchange of two particles.

- The Δ^{++} is the lightest charge-2 state $\rightarrow$ it is expected to be in the ground state without orbital angular momentum, so the position-space wave-function is symmetric. (The vanishing orbital momentum is supported by the value of the magnetic moment predicted in the quark model).

- The spin-wave function for a Spin 3/2 state is symmetric: $|\uparrow\uparrow\uparrow\rangle$

$\Rightarrow$ postulate of additional quantum number "colour": $|q_i\rangle$, $i = 1, 2, 3$ ("red, green, blue")

- The Pauli principle can be satisfied in the Baryon wave-functions are antisymmetric in colour

$$|\psi\rangle \sim \epsilon^{ijk} |q_i, q_j, q_k\rangle \tag{2.20}$$

Colour-$SU(3)$

Since the colour quantum number is not observed, one expects that the interactions of quarks are invariant under **rotations in colour space**,

$$|q_i\rangle \to U_i^{\ j} |q_j\rangle \, . \tag{2.21}$$

The matrices U are complex three-by three matrices since the wave-functions $|q_i\rangle$ are complex fields. We have introduced the convention to sum over equal upper and lower indices as in relativity. Since symmetry transformations in QM are implemented by unitary transformations[2] we demand

$$U^\dagger = U^{-1} \tag{2.22}$$

- The set of unitary $N \times N$ matrices forms a **group** called $U(N)$. A matrix in $U(N)$ has $2N^2 - N^2 = N^2$ independent, real, elements.
- The group of matrices that in addition have $\det U = 1$ is called $SU(N)$. A matrix in $SU(N)$ has $N^2 - 1$ independent, real elements. Therefore an $SU(3)$ transformation is described by eight real parameters.

The matrices in $U(3)$ and $SU(3)$ are examples of a **group** G since they satisfy the axioms

- $f, g \in G$, there is a product $\circ$ such that $f \circ g = h \in G$.
- The product is associative: $f \circ (g \circ h) = (f \circ g) \circ h$.
- There exists a unit element $e \in G$: $e \circ g = g \circ e = g$ for all $g \in G$.
- For every element $g \in G$ there is an inverse g^{-1} satisfying $g^{-1} \circ g = g \circ g^{-1} = e$.

Groups of noncommuting elements,

$$f \circ g \neq g \circ f \tag{2.23}$$

are called **non-abelian**.

Confinement

The Baryon wave-functions (2.20) transform under colour rotations according to

$$|\psi\rangle = \epsilon^{ijk} |q_i, q_j, q_k\rangle = \epsilon^{ijk} U_i^{\ i'} U_j^{\ j'} U_k^{\ k'} |q_{i'}, q_{j'}, q_{k'}\rangle = \det U \epsilon^{ijk} |q_i, q_j, q_k\rangle = \det U |\psi\rangle \tag{2.24}$$

$\Rightarrow$ The baryon states are invariant under $SU(3)$ transformations. They are called “colour singlets”. This explains the name “colour” since a “white” object is obtained from combining “red”, “green” and “blue” quarks.

- To explain the non-observation of free quarks, or of composite states other than the baryons and mesons, one then postulates that only colour-singlets are observable, i.e. coloured objects are always **confined** in colour neutral bound states. This property should be explained by the theory of quark interactions. QCD is consistent with confinement, although it has not rigorously proven yet

[2] anti-unitary transformations are also allowed but only relevant for the time conjugation transformation

- In addition to the quarks with the transformation (4.2) , we have anti-quarks that transform with the complex conjugate transformation.

$$|\bar{q}^i\rangle \to |\bar{q}^j\rangle \, (U^\dagger)_j{}^i \tag{2.25}$$

Objects transforming with the conjugate transformations are denoted with an upper index.

- Colour-singlet wave-functions for meson (i.e. quark-antiquark) states are obtained by contracting upper and lower indices using a Kronecker-delta:

$$|\phi\rangle \sim \delta_i{}^j \, |\bar{q}^i q_j\rangle \to (U^\dagger U)_k{}^l \, |\bar{q}^k q_l\rangle = \delta_k{}^l \, |\bar{q}^k q_l\rangle \tag{2.26}$$

Evidence of colour: $e^+e^- \to$ **hadrons**

The so-called *R-ratio* is defined as

$$R = \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}, \tag{2.27}$$

where $\sigma(e^+e^- \to \text{Hadrons})$ is the total production cross section for hadronic final states in electron-positron collisions. In a first approximation, the cross section $\sigma(e^+e^- \to \text{Hadrons})$ can be computed in terms of the production cross section of quarks, $\sigma(e^+e^- \to q\bar{q})$ in QED (see Chapter 3). The production cross section of a quark-antiquark pair $q\bar{q}$ is proportional to the muon-production cross section up to the electric charge Q_q and the multiplicity N_c due to the colour quantum number:

$$\sigma(e^+e^- \to q\bar{q}) = N_c Q_q^2 \sigma(e^+e^- \to \mu^+\mu^-) \tag{2.28}$$

The experimental results [14] are compatible with the expectations for $N_c = 3$ colours:

$E_{CM} < 2.5$ **GeV:** production of u,d,s quarks:

$$\begin{aligned} R_{\text{quark}} &= N_c((\tfrac{2}{3})^2 + (\tfrac{1}{3})^2 + (\tfrac{1}{3})^2 = N_c \frac{2}{3} = 2 \\ R_{\text{exp}} &\approx 2.2 \end{aligned} \tag{2.29}$$

4 **GeV** $< E_{CM} < 9$ **GeV:** production of u,d,s,c quarks:

$$\begin{aligned} R_{\text{quark}} &= N_c 2((\tfrac{2}{3})^2 + (\tfrac{1}{3})^2) = N_c \frac{10}{9} = 3.33\ldots \\ R_{\text{exp}} &\approx 3.6 \end{aligned} \tag{2.30}$$

11 **GeV** $< E_{CM} < 90$ **GeV:** production of u,d,s,c,b quarks:

$$\begin{aligned} R_{\text{quark}} &= N_c(2(\tfrac{2}{3})^2 + 3(\tfrac{1}{3})^2) = N_c \frac{11}{9} = 3.66\ldots \\ R_{\text{exp}} &\approx 4 \end{aligned} \tag{2.31}$$

- The systematic deviation $R_{\text{exp}}/R_{\text{quark}} \approx 1.1$ is due to missing higher-order corrections in perturbation theory that can be computed in QCD.
- For energies near $E_{CM} \sim 2m_q$ for any of the quark flavours, bound states can form (e.g. the $c\bar{c}$-bound state J/ψ, the $b\bar{b}$ bound state Υ) and the description in terms of the initial production of free, highly energetic quarks is not valid.

2.4 Towards QCD

To summarize the picture of hadron substructure and quarks obtained so far:

- hadrons are composed of quarks, which have electric charges $+\frac{2}{3}$, $-\frac{1}{3}$ and a colour-quantum number.
- interactions among quarks at low energies are so strong that only colour-singlet states are observed as free particles.
- scattering at high energies (e.g. DIS and $e^+e^- \to$ hadrons) can be described in terms of very light, quasi-free partons

To explain these features, the interaction of quarks must have some peculiar properties:

- it must couple to the colour quantum number
- it must be weak at high energies ("asymptotic freedom") but very strong at low energies ("infrared slavery")

At the time when the quark model and the parton model were proposed, no known Quantum field theory was in agreement with these properties, in particular QED has the property to grow stronger at high energies. Also theories where the force among quarks is due to a scalar particle were ruled out. The breakthrough was the discovery of Gross/Politzer and Wilzcek (1973) that a class of theories is consistent with asymptotic freedom at NLO in perturbation theory. These so-called "nonabelian gauge theories" or "Yang-Mills theories" describe massless vector bosons (analogous to the photon) with self-interactions. Therefore it was proposed that the interaction of quarks is carried by massless vector bosons ("gluons"), that themselves carry colour quantum numbers. The theory called Quantum Chromodynamics is invariant under $SU(3)$ transformations, so it is a "$SU(3)$ gauge theory".

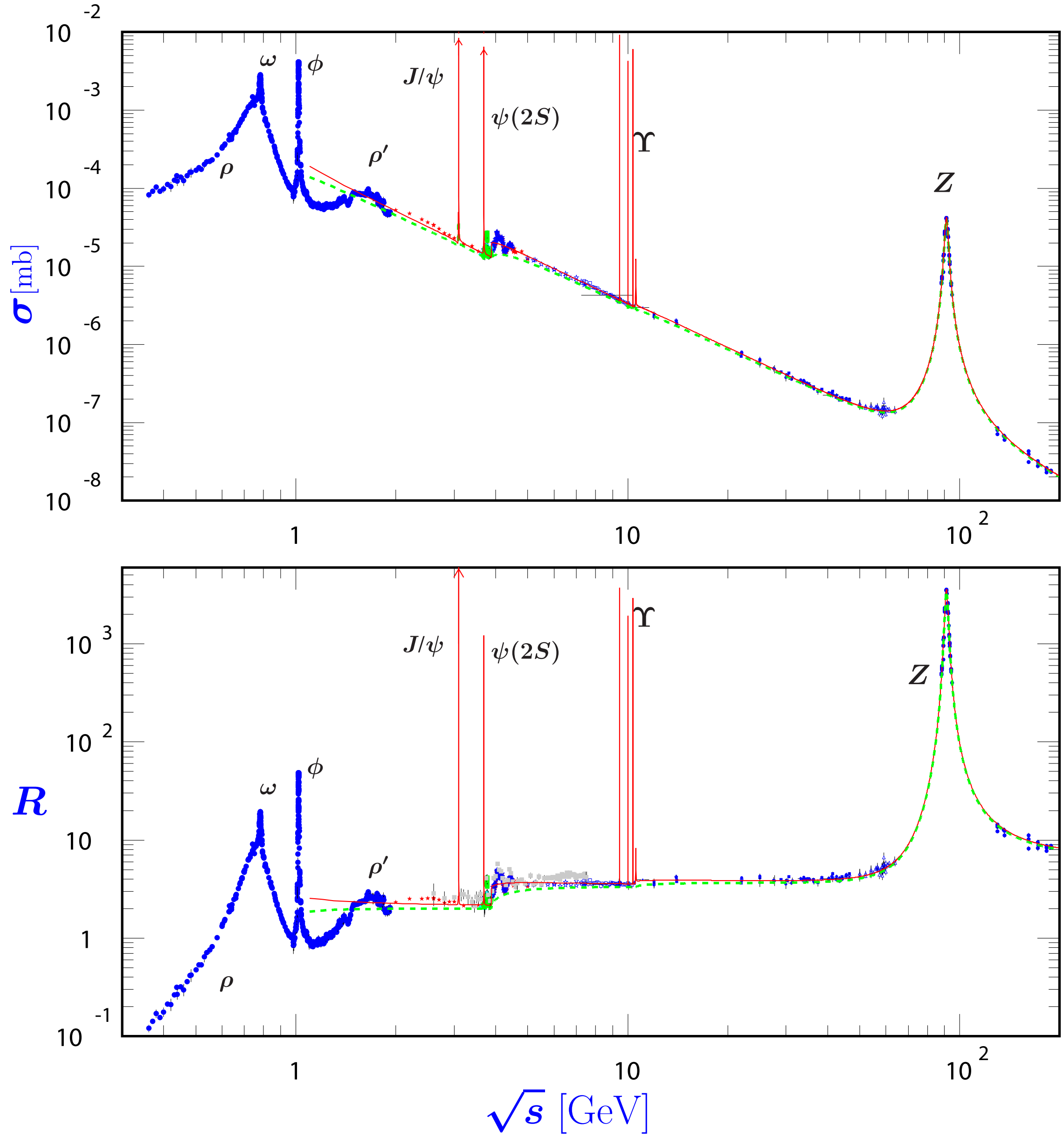


Figure 2.1: Experimental result for the cross section $e^+e^- \to$ hadrons and for the R-ratio (10.112). Taken from [14].

Chapter 3

Basics of QFT and QED

- Objective: describe scattering processes of relativistic particles
- Requires description consistent with quantum mechanics and relativity
- Relativistic one-particle states $|m, p, s, \ldots\rangle$: characterized by
  - mass m
  - momentum $\vec{p}$, energy $E = p^0 \sqrt{m^2 + |\vec{p}|^2}$
  - "helicity" s = projection of spin on momentum
  - other quantum numbers: electric charge, colour state...

  The properties of relativistic one-particle states will be reviewed in Section 3.1
- A general prediction of the combination of relativity and quantum mechanics is the presence of **antiparticles**: for every species of particles with mass m, electric charge Q, there exists a species of antiparticles with identical properties, but charge $-Q$. (e.g. positrons as antiparticles as electrons.) Historically, this was discovered because it is impossible to consistently omit contributions to wavepackets with the negative solutions $E = -\sqrt{m^2 + |\vec{p}|^2}$. These solutions can be re-interpreted as antiparticles with positive energies. One can show in general that antiparticles are required in order to satisfy causality in a relativistic quantum theory.
- Because of the existence of antiparticles, particle number is not a conserved quantum number, e.g. processes like $e^- e^+ \to \gamma$ are possible. This is one of the reasons why a naive extension of single-particle quantum mechanics to the relativistic case is not consistent.

3.1 Quantum numbers of relativistic particles

Later on we will calculate scattering amplitudes in QCD for helicity eigenstates. Therefore we here give an overview over the definition of helicity and the transformations of helicity states under Poincaré transformations. For more details see [4, 5]

3.1.1 Poincaré group

Quantum states of particles are characterized by their behaviour under **Poincaré** transformations. A general Poincaré transformation g is a combination of a Lorentz transformation Λ and a space-time translation by a four-vector a^μ:

$$g(\Lambda, a) : x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu - a^\mu. \tag{3.1}$$

The homogeneous Lorentz transformation Λ is defined by the condition

$$g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \overset{!}{=} g_{\rho\sigma} \tag{3.2}$$

Lorentz transformations can be constructed by successive **infinitesimal transformations**

$$\Lambda^\mu{}_\nu(\delta\omega) = \delta^\mu_\nu + \delta\omega^\mu{}_\nu + \dots . \tag{3.3}$$

The condition for a Lorentz transformation becomes

$$\begin{aligned} g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma &= g_{\mu\nu}(\delta^\mu_\rho + \delta\omega^\mu{}_\rho)(\delta^\nu_\sigma + \delta\omega^\nu{}_\sigma) + \dots \\ &= g_{\rho\sigma} + \delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma} + \dots \overset{!}{=} g_{\rho\sigma}, \end{aligned} \tag{3.4}$$

so that the $\delta\omega$ are antisymmetric:

$$\delta\omega_{\sigma\rho} = -\delta\omega_{\rho\sigma}. \tag{3.5}$$

Since antisymmetric four-by-four matrices have six independent entries, this shows that Lorentz transformations can be parametrized by six parameters, consistent with the discussion above. To use the antisymmetry of the $\delta\omega_{\sigma\rho}$ we have to lower one index in (3.3) so we can write this expression as

$$\Lambda^\mu{}_\nu(\delta\omega) = \delta^\mu_\nu + \delta\omega_{\alpha\beta}\, g^{\alpha\mu}\, \delta^\beta_\nu \equiv \delta^\mu_\nu - \frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}(M^{\alpha\beta})^\mu{}_\nu, \tag{3.6}$$

with the generators of infinitesimal Lorentz transformations

$$(M^{\alpha\beta})^\mu{}_\nu = \mathrm{i}(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu). \tag{3.7}$$

The generators of rotations and boosts can be identified as

$$K^i = M^{0i}\,, \qquad J^k = \frac{1}{2}\epsilon^{ijk}M^{ij}\,, \tag{3.8}$$

for instance $J^3 = M^{12}$. A finite transformation can be constructed by exponentiation:

$$\Lambda(\omega) = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right). \tag{3.9}$$

A representation of the operators M and P in terms of differential operators can be obtained by considering the transformation of a scalar field $\phi(x)$:

$$\phi'(x) = \phi(\Lambda^{-1}(x+a)) \equiv e^{-\frac{\mathrm{i}}{2}\omega_{\alpha\beta}L^{\alpha\beta}-\mathrm{i}a_\alpha P^\alpha}\phi(x). \tag{3.10}$$

Considering infinitesimal transformations,

$$\begin{aligned}
\phi(x+\delta x) &= \phi(x) + \delta x^\mu \partial_\mu \phi(x) + \dots \\
&\equiv (1 - \mathrm{i}\delta x^\mu P_\mu + \dots)\phi(x) \\
\phi(x - \tfrac{i}{2}\delta\omega_{\alpha\beta}M^{\alpha\beta}x) &= \phi(x) + \tfrac{i}{2}\delta\omega_{\alpha\beta}(M^{\alpha\beta})^\mu{}_\nu x^\nu \partial_\mu\phi(x) + \dots \\
&\equiv (1 - \frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}L^{\alpha\beta} + \dots)\phi(x)
\end{aligned} \tag{3.11}$$

one finds the representation

$$L^{\alpha\beta} = \mathrm{i}\left(x^\alpha\partial^\beta - x^\beta\partial^\alpha\right), \qquad P^\alpha = \mathrm{i}\partial^\alpha \tag{3.12}$$

These operators satisfy the following commutation relations

$$[L^{\mu\nu}, L^{\rho\sigma}] = -\mathrm{i}\left(g^{\mu\rho}L^{\nu\sigma} - g^{\mu\sigma}L^{\nu\rho} - g^{\nu\rho}L^{\mu\sigma} + g^{\nu\sigma}L^{\mu\rho}\right) \tag{3.13}$$

$$[L^{\alpha\beta}, P^\gamma] = -\mathrm{i}(g^{\alpha\gamma}P^\beta - g^{\beta\gamma}P^\alpha), \qquad [P^\alpha, P^\beta] = 0 \tag{3.14}$$

called the **Poincaré Algebra**.

The generators of boosts and rotations are explicitly given by

$$J^1 = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-\mathrm{i}\\ 0&0&\mathrm{i}&0 \end{pmatrix}, \qquad J^2 = \begin{pmatrix} 0&0&0&0\\ 0&0&0&\mathrm{i}\\ 0&0&0&0\\ 0&-\mathrm{i}&0&0 \end{pmatrix}, \qquad J^3 = \begin{pmatrix} 0&0&0&0\\ 0&0&-\mathrm{i}&0\\ 0&\mathrm{i}&0&0\\ 0&0&0&0 \end{pmatrix}, \tag{3.15}$$

$$K^1 = \begin{pmatrix} 0&\mathrm{i}&0&0\\ \mathrm{i}&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \qquad K^2 = \begin{pmatrix} 0&0&\mathrm{i}&0\\ 0&0&0&0\\ \mathrm{i}&0&0&0\\ 0&0&0&0 \end{pmatrix}, \qquad K^3 = \begin{pmatrix} 0&0&0&\mathrm{i}\\ 0&0&0&0\\ 0&0&0&0\\ \mathrm{i}&0&0&0 \end{pmatrix} \tag{3.16}$$

They satisfy the commutation relations

$$[J^i, J^j] = \mathrm{i}\epsilon^{ijk}J^k \qquad \text{(relations of angular momentum)} \tag{3.17}$$

$$[J^i, K^j] = \mathrm{i}\epsilon^{ijk}K^k \qquad (\vec{K}\text{ transforms as 3-vector operator)} \tag{3.18}$$

$$[K^i, K^j] = -\mathrm{i}\epsilon^{ijk}J^k. \tag{3.19}$$

3.1.2 Relativistic one-particle states

As usual, quantum states are characterized by their eigenvalues of a complete set of commuting observables. Out of the generators of the Poincaré group, one can form the operator

$$P^2 = g_{\alpha\beta}P^\alpha P^\beta \tag{3.20}$$

that commutes with P^μ and $L^{\alpha\beta}$. This can be checked explicitly and should also be intuitively clear since P^2 is a Lorentz-scalar. The eigenvalue of P^2 is the mass m^2.

A second commuting operator can be constructed using the **Pauli-Lubanski-Vector**

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma} \tag{3.21}$$

that satisfies the commutator relations characteristic of a four vector:

$$[W^\mu, P^\nu] = 0, \tag{3.22}$$

$$[W^\mu, M^{\rho\sigma}] = \mathrm{i}\left(g^{\mu\rho}W^\sigma - g^{\mu\sigma}W^\rho\right). \tag{3.23}$$

As for P^2, it follows that the operator W^2 commutes with all generators of the Poincaré group. In order to interpret W^2, we use that it can be evaluated in any reference frame since it is a Lorentz scalar.

Massive particles: For massive particles it is possible to go to the rest frame

$$p_0^\mu = (m, \vec{0}). \tag{3.24}$$

Therefore in the restframe

$$\begin{aligned} W_0 &= 0 \\ W_i &= \frac{m}{2}\epsilon_{ijk}L^{jk} = -mJ_i \end{aligned} \tag{3.25}$$

i.e. for a massive particle the Pauli-Lubanski vector reduces to the angular momentum operator in the rest-frame. Therefore

$$W^2 = -m^2\vec{J}^2 \tag{3.26}$$

From this property follows the classification of massive one-particle states:

- The spin quantum numbers of a massive relativistic particle are identical to the non-relativistic case, i.e. the eigenvalues of $\vec{J}^2$ are $s(s+1)$ with $s = 0, \frac{1}{2}, 1 \ldots$.
- Since $[P^i, J^l] = 0$ for states with momentum p_0^μ, the states in the rest-frame can be chosen as eigenstates of one component of $\vec{J}$, e.g. the z-component, with eigenvalues $s_z = -s, -s+1, \ldots s-1, S$.
- the operators $\vec{J}$ are represented on the spin s-states by the same matrices as in non-relativistic QM, e.g.

$$s = \frac{1}{2}: \quad J^i = \frac{\sigma^i}{2} \qquad\qquad s = 1: \quad (J^i)_{jk} = -\mathrm{i}\epsilon^{ijk} \tag{3.27}$$

- In a general Lorentz frame, one can consider the projection

$$W_n = n \cdot W \tag{3.28}$$

on some space-like unit vector n, $n^2 = -1$ that is orthogonal to p^μ.

- A convenient choice is **"helicity"**, the projection of the spin on the momentum:

$$h = \frac{\vec{p}\cdot\vec{J}}{|\vec{p}|} \tag{3.29}$$

This can be written in the form (3.28) using the space-like unit vector

$$n_p^\mu = \frac{p^0}{m}\begin{pmatrix}\frac{|\vec{p}|}{p^0}\\ \frac{\vec{p}}{|\vec{p}|}\end{pmatrix} = \frac{p^0}{m|\vec{p}|}p^\mu - \frac{m}{|\vec{p}|}g^{\mu 0}, \tag{3.30}$$

$$h = \frac{n_p\cdot W}{m} = -\frac{1}{|\vec{p}|}W_0 = \frac{1}{2|\vec{p}|}\epsilon_{0ijk}p^i L^{jk} = \frac{\vec{p}\cdot\vec{J}}{|\vec{p}|} \tag{3.31}$$

To summarize, the massive, relativistic one-particle states satisfy the eigenvalue equations

$$P^2\,|m,p,s,s_p,\ldots\rangle = m^2\,|m,p,s,s_p,\ldots\rangle \tag{3.32}$$

$$P^\mu\,|m,p,s,s_p,\ldots\rangle = p^\mu\,|m,p,s,s_p,\ldots\rangle\ ,\quad p^0=\sqrt{\vec{p}^2+m^2} \tag{3.33}$$

$$W^2\,|m,p,s,s_p,\ldots\rangle = m^2 s(s+1)\,|m,p,s,s_p,\ldots\rangle \tag{3.34}$$

$$h\,|m,p,s,s_p,\ldots\rangle = s\,|m,p,s,s_p\ldots\rangle\ ,\quad h=\frac{\vec{p}\cdot\vec{J}}{|\vec{p}|} \tag{3.35}$$

The Poincarè transformations are represented on the one-particle states by unitary operators $U(\Lambda,a)$. Their action on the states can be shown to take the form

$$U(1,a)\,|\ldots p,\ldots\rangle = e^{-ia\cdot p}\,|\ldots,p,\ldots\rangle \tag{3.36}$$

$$U(\Lambda,0)\,|p,s,s_p,\ldots\rangle = |\Lambda p,s,s_p',\ldots\rangle\,D^s_{s_p' s_p}(W) \tag{3.37}$$

where the matrix D^s is a spin S-representation of a rotation $W(\Lambda,p)$ (the so-called "Wigner rotation"). The explicit form is determined from Λ and p in such a way that the rest-frame momentum p^0 is left invariant, $Wp^0=p^0$ and will not be needed in this lecture.

The group of transformations that leaves the momentum of the particle invariant is called the **little group**. In the rest frame, the momentum is invariant under spatial rotations, i.e. the little group is $SO(3)$.

The state of a massive particle with momentum p is defined by a boost from the rest frame:

$$|p,s,s_p\rangle = U(L_p)\,|p^0,s,s_p\rangle \tag{3.38}$$

where L_p is the boost from p_0 to p and $U(L)$ is the unitary operator implementing the Lorentz transformation. To obtain the transformation law under a general Lorentz transformation Λ one notes that

$$\Lambda L_p = L_{\Lambda p}\underbrace{L_{\Lambda p}^{-1}\Lambda L_p}_{\equiv W(\Lambda,p)} \tag{3.39}$$

where $L_{\Lambda p}$ is the Lorentz boost from p_0 to Λp. The transformation W transforms the momentum p^0 to $p^0 \to p \to \Lambda p \to p^0$, i.e. it leaves the momentum invariant. Therefore it is part of the little group, i.e. a rotation. The Lorentz transformation of the general one-particle state is then obtained as

$$U(\Lambda)\,|p,s,s_p\rangle = U(\Lambda)U(L_p)\,|p^0,s,s_p\rangle = U(L_{\Lambda p})D^s_{s'_p s_p}(W) = |\Lambda p, s, s'_p\rangle\, D^s_{s'_p s_p}(W) \tag{3.40}$$

where $D^s_{s'_p s_p}(W) = \langle s, s'_p|U(W)|s, s_p\rangle$ is the matrix representing the Wigner rotation on the spin s space.

Massless particles For massless particles it is possible to go a frame where

$$\bar{p}^\mu = \bar{p}(1,0,0,1) \tag{3.41}$$

The Pauli-Lubanski vector becomes

$$\begin{aligned} W_\mu &= -\frac{1}{2}\bar{p}(\epsilon_{\mu 0\rho\sigma} + \epsilon_{\mu 3\rho\sigma})M^{\rho\sigma} \\ &= \bar{p}\begin{pmatrix} \epsilon^{0312}L^{12} \\ (\epsilon^{1023}L^{23} + \epsilon^{1302}L^{02}) \\ (\epsilon^{2031}L^{31} + \epsilon^{2301}L^{01}) \\ \epsilon^{3012}L^{12} \end{pmatrix} = \bar{p}\begin{pmatrix} J^3 \\ -(J^1 + K^2) \\ -J^2 + K^1 \\ -J^3 \end{pmatrix} \equiv \bar{p}\begin{pmatrix} J^3 \\ -A \\ -B \\ -J^3 \end{pmatrix} \end{aligned} \tag{3.42}$$

with

$$A = J_1 + K_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \qquad B = J_2 - K_1 = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} \tag{3.43}$$

- The angular momentum operator J_3 generates rotations around the z-axis, i.e. the group $SO(2)$ of rotation in two dimensions. These obviously leave the momentum invariant and therefore belong to the little group.
- The full little group is generated by J^3, A and B.
- Only the $SO(2)$ part is relevant for the classification of massless particle states since it is possible to show that non-vanishing eigenvalues of A and B give rise to representations with continuous spin that are not observed in nature [4, 5].
- On the physically relevant state-space, the Pauli-Lubanski vector satisfies

$$W^2 = 0\,, \qquad W\cdot P = 0 \tag{3.44}$$

This can be shown to imply that the Pauli-Lubanski vector is proportional to the momentum:

$$W^\mu = hP^\mu \tag{3.45}$$

For the frame where the momentum is along the z-axis (3.41), this can also be seen from the explicit expression.

The commutation relations of W^μ imply that $[h, P^\mu] = [h, M^{\mu\nu}] = 0$

$\Rightarrow$ h is a Casimir operator. The states of a massless particle with momentum $\vec{p}$, are again characterized by the eigenvalues of the helicity

$$h = \frac{\vec{p} \cdot \vec{J}}{|\vec{p}|} \tag{3.46}$$

- The non-trivial eigenvalue equations of the massless one-particle states are

$$P^\mu |p, s, \ldots\rangle = p^\mu |p, s, \ldots\rangle \ , \quad p^0 = |\vec{p}| \tag{3.47}$$
$$h |p, s, \ldots\rangle = s |p, s, s \ldots\rangle \ , \tag{3.48}$$

- The value of the helicity s is quantized,

$$s = 0, \pm\frac{1}{2}, \pm 1, \ldots \tag{3.49}$$

In contrast to them massive case, this doesn't follow from the angular momentum algebra, since only J_3 is relevant. The allowed values of s instead follow from topological properties of the Poincaré group. Recall that in non-relativistic quantum mechanics, for particles with half-integer spin a rotation with $\theta = 2\pi$ can turn a state into its negative, while a rotation with $\theta = 4\pi$ is the unity transformation. This turns out to be true for the relativistic case as well.

The action of Poincaré transformations on the states is given by

$$U(1, a) |p, s, \ldots\rangle = e^{-ia\cdot p} |p, s, \ldots\rangle \tag{3.50}$$
$$U(\Lambda, 1) |p, s, \ldots\rangle = e^{i\theta(\Lambda,p)s} |\Lambda p, s, \ldots\rangle \tag{3.51}$$

where the explicit form of the angle $\theta(\Lambda, p)$ is not relevant for us.

- The Lorentz transformation does not mix different values of s (intuitively: it is not possible to "overtake" a massless particle and change the sign of the projection of its spin on the momentum).
- A parity transformation transforms $\vec{p} \to -\vec{p}$, $\vec{J} \to \vec{J}$ and therefore transforms the states $s \to -s$. In a parity invariant theory such as QED both states $\pm s$ must be present (i.e. left-and right-handed electrons in the approximation $m_e \to 0$. The weak interactions violate parity, so if neutrino masses are neglected, it is possible to have only left-handed neutrinos.
- CPT invariance requires that for each particle with helicity s there is an antiparticle with helicity $-s$.

The one-particle state with momentum p is defined in analogy to (3.38) by the Lorentz boost of the state with the momentum $\bar{p}$ (3.41). The relevant part of the little-group transformation $W(\Lambda, p) = L^{-1}_{\Lambda p} \Lambda L_p$ defines a rotation in the x_1-x_2-plane that can be parametrized by an angle $\theta(\Lambda, p)$. The action of the unitary operator representing this rotation is given by

$$U(W) |\bar{p}, s\rangle = e^{\mathrm{i}\theta J_3} |\bar{p}, s\rangle = e^{\mathrm{i}\theta s} |\bar{p}, s\rangle \tag{3.52}$$

if the states $|\bar{p}, s\rangle$ are chosen as eigenstates of J_3.

3.2 Quantum fields

Relation to fields: field operators create/annihilate particles

3.2.1 Scalar fields

Spin 0 particle: scalar field $\phi(x)$. Example, complex scalar:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0} \left(a(\vec{p}) e^{-ipx} + b^\dagger(\vec{p}) e^{ipx} \right) |_{p^0 = \sqrt{\vec{p}^2 + m^2}} \tag{3.53}$$

Solution to Klein Gordon equation

$$(\Box + m^2)\phi(x) = 0 \tag{3.54}$$

a, b: operators with commutation relations

$$[a_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{p})] = [b_\lambda(\vec{k}), b^\dagger_{\lambda'}(\vec{p})] = \delta_{\lambda,\lambda'}(2\pi)^3(2p^0)\delta^3(\vec{k} - \vec{p}) \tag{3.55}$$

Annihilation/creation operators for scalar particle ϕ^-, antiparticle ϕ^+. Example: two-particle state

$$|\phi^-_{k_1}, \phi^+_{k_2}\rangle = a^\dagger(\vec{k}_1) b^\dagger(\vec{k}_2) |0\rangle \tag{3.56}$$

Bose Symmetry: states symmetric

$$|\phi^-_{k_1}, \phi^-_{k_2}\rangle = |\phi^-_{k_2}, \phi^-_{k_1}\rangle \tag{3.57}$$

Propagator:

$$\mathrm{i}D_F(x, y) = \langle 0|T[[\phi(x)\phi^\dagger(y)]]|0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip(x-y)} D_F(p^2) \tag{3.58}$$

with

$$D_F(p^2) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} \tag{3.59}$$

Time ordered product:

$$T\left[\phi(x)\phi^\dagger(y)\right] = \phi(x)\phi^\dagger(y)\theta(x^0 - y^0) + \phi^\dagger(y)\phi(x)\theta(y^0 - x^0). \tag{3.60}$$

3.2.2 Spinor fields

Massive spin one-half particle: four-component Dirac-spinor field

$$\psi(x) = \sum_{\lambda = L,R} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0} \left(b_\lambda(\vec{p}) u_\lambda(p) e^{-ipx} + d^\dagger_\lambda(\vec{p}) v_\lambda(p) e^{ipx} \right) \tag{3.61}$$

Dirac spinors

Spinors u, v: solution to Dirac equation in momentum space:

$$(\not{p} - m)u_\lambda(p) = 0 \tag{3.62a}$$
$$(\not{p} + m)v_\lambda(p) = 0 \tag{3.62b}$$

with

$$\not{p} = \gamma^\mu p_\mu \tag{3.63}$$

with gamma matrices that satisfy

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{3.64}$$

Explicit form (in "chiral representation")

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \tag{3.65}$$

with

$$\sigma^\mu = (\mathbf{1}, \vec{\sigma})\ , \qquad \bar{\sigma}^\mu = (\mathbf{1}, -\vec{\sigma}) \tag{3.66}$$
$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\ , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.67}$$

Note that different conventions are also used, for instance in [8]

Spinors can be chosen as helicity eigenstates:

$$h = \frac{\vec{p} \cdot \vec{S}}{|\vec{p}|} \qquad \vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{3.68}$$
$$h\, u_{R/L}(p) = \pm\frac{1}{2} u_{R/L}(p) \tag{3.69}$$
$$h\, v_\lambda(p) = \lambda v_\lambda(p) \tag{3.70}$$
$$\tag{3.71}$$

Spinors satisfy

$$\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = \bar{v}_\sigma(p)\gamma^\mu v_{\sigma'}(p) = 2p^\mu\, \delta_{\sigma\sigma'} \tag{3.72}$$

where the conjugate spinor is defined as

$$\bar{u} = u^\dagger \gamma^0 \tag{3.73}$$

Massless spin one-half particles

Dirac equation decouples into two-component equations:

$$\begin{pmatrix} 0 & p_\mu \bar{\sigma}^\mu \\ p_\mu \sigma^\mu & \end{pmatrix} \begin{pmatrix} u_+(p) \\ u_-(p) \end{pmatrix} = \begin{pmatrix} 0 & p^0 + \vec{p}\cdot\vec{\sigma} \\ p^0 - \vec{p}\cdot\vec{\sigma} & \end{pmatrix} \begin{pmatrix} u_+(p) \\ u_-(p) \end{pmatrix} \tag{3.74}$$

$\Rightarrow$ upper and lower components are helicity eigenstates:

$$\frac{\vec{p}\cdot\vec{\sigma}}{p^0} u_\pm(p) = \pm u_\pm(p) \tag{3.75}$$

Dirac spinors for helicity $\pm$:

$$u_R(p) = \begin{pmatrix} u_+(p) \\ 0 \end{pmatrix} \qquad\qquad u_L(p) = \begin{pmatrix} 0 \\ u_-(p) \end{pmatrix} \tag{3.76}$$

Find explicit solutions of two-component equations

$$p_\mu \sigma^\mu u_+(p) = \begin{pmatrix} p^0 - p^3 & -(p^1 - \mathrm{i}p^2) \\ -(p^1 + \mathrm{i}p^2) & p^0 + p^3 \end{pmatrix} u_+(p) = 0 \tag{3.77}$$

$$p_\mu \bar{\sigma}^\mu u_-(p) = \begin{pmatrix} p^0 + p^3 & p^1 - \mathrm{i}p^2 \\ p^1 + \mathrm{i}p^2 & p^0 - p^3 \end{pmatrix} u_-(p) = 0 \tag{3.78}$$

with $p^2 = 0$.

Normalizing the solutions according to

$$u_\pm^\dagger(p) u_\pm(p) = 2p^0 \,, \qquad\qquad u_\pm^\dagger(p) u_\mp(p) = 0 \tag{3.79}$$

the spinors can be written, up to a phase depending on conventions

$$u_+ = \frac{1}{\sqrt{(p^0 - p^3)}} \begin{pmatrix} p^1 - \mathrm{i}p^2 \\ p^0 - p^3 \end{pmatrix} = \sqrt{2p^0} \begin{pmatrix} e^{-\mathrm{i}\varphi} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \tag{3.80}$$

$$u_- = \frac{1}{\sqrt{(p^0 - p^3)}} \begin{pmatrix} p^0 - p^3 \\ -(p^1 + \mathrm{i}p^2) \end{pmatrix} = \sqrt{2p^0} \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{\mathrm{i}\varphi} \cos\frac{\theta}{2} \end{pmatrix} \tag{3.81}$$

where the momentum is parametrized in terms of angular coordinates:

$$p^\mu = p^0(1, \cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta) \tag{3.82}$$

The solution in [9] is related to the one above by a phase:

$$\tilde{u}_+ = \frac{1}{\sqrt{(p^0 + p^3)}} \begin{pmatrix} p^0 + p^3 \\ p^1 + \mathrm{i}p^2 \end{pmatrix} = \sqrt{2p^0} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{\mathrm{i}\varphi} \sin\frac{\theta}{2} \end{pmatrix} = e^{\mathrm{i}\varphi} u_+ \tag{3.83}$$

Spinor products

Out of spinors for particles with different momenta we can form the two "scalar products"

$$
\begin{aligned}
\langle pk\rangle &\equiv \bar{u}_L(p)u_R(k) = u_-^\dagger(p)u_+(k) \\
&= \frac{1}{\sqrt{(p^0-p^3)}\sqrt{(k^0-k^3)}}\left[(p^0-p^3)(k^1-\mathrm{i}k^2)-(k^0-k^3)(p^1-\mathrm{i}p^2)\right] \\
&= -\langle kp\rangle \\
[pk] &\equiv \bar{u}_R(p)u_L(k) = u_+^\dagger(p)u_-(k) \\
&= \frac{1}{\sqrt{(p^0-p^3)(k^0-k^3)}}\left[(p^1+\mathrm{i}p^2)(k^0-k^3)-(p^0-p^3)(k^1+\mathrm{i}k^2)\right] \\
&= -[kp]
\end{aligned}
$$

For real momenta the spinor products satisfy

$$\langle pk\rangle^* = [kp] \tag{3.84}$$

These spinor product will be a basic building block to express scattering amplitudes in the "spinor-helicity" method to be discussed in Part II of the lecture.

Quantization

Creation and annihilation operators satisfy anticommutation relations

$$\{b_\lambda(\vec{k}), b^\dagger_{\lambda'}(\vec{p})\} = \delta_{\lambda,\lambda'}(2\pi)^3(2p^0)\delta^3(\vec{k}-\vec{p}) \tag{3.85}$$

$$\{d_\lambda(\vec{k}), d^\dagger_{\lambda'}(\vec{p})\} = \delta_{\lambda,\lambda'}(2\pi)^3(2p^0)\delta^3(\vec{k}-\vec{p}) \tag{3.86}$$

Fermi Symmetry: states anti-symmetric

$$|\psi^{\lambda_1}_{k_1}, \psi^{\lambda_2}_{k_2}\rangle = -|\psi^{\lambda_2}_{k_2}, \psi^{\lambda_1}_{k_1}\rangle \tag{3.87}$$

Propagator:

$$\mathrm{i}S_F(x-y) = \langle 0|T[[\psi(x)\bar{\psi}(y)]]|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ik(x-y)} S_F(p^2) \tag{3.88}$$

with

$$S_F(p^2) = \frac{\mathrm{i}(\not{p}+m)}{p^2-m^2+\mathrm{i}\epsilon} \tag{3.89}$$

3.2.3 Massless vector bosons

Mode decomposition of field:

$$A^\mu(x) = \sum_\lambda \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\left(a_\lambda(\vec{p})\epsilon^\mu_\lambda(p)e^{-ipx} + a^\dagger_\lambda(\vec{p})\epsilon^{\mu,*}_\lambda(p)e^{ipx}\right) \tag{3.90}$$

Commutation relations

$$[a_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{p})] = \delta_{\lambda,\lambda'}(2\pi)^3(2p^0)\delta^3(\vec{k}-\vec{p}) \tag{3.91}$$
$$\tag{3.92}$$

Polarization vectors and gauge invariance

Polarization vectors

$$(\epsilon_\lambda(p)\cdot\epsilon^*_{\lambda'}(p)) = -\delta_{\lambda\lambda'} \tag{3.93}$$

Gauge choice

$$\partial_\mu A^\mu = 0\ , \qquad p_\mu\epsilon^\mu = 0 \tag{3.94}$$

Polarization vectors determined up to gauge transformation:

$$\epsilon^\mu(p) \to \epsilon^\mu(p) + ap^\mu \tag{3.95}$$

In the frame where $p^\mu = (p, 0, 0, p)$ the polarization vectors can be chosen as

$$\epsilon^\mu_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\\pm\mathrm{i}\\0\end{pmatrix} \tag{3.96}$$

They are helicity Eigenstates with eigenvalues $s=\pm1$ since they in the chosen frame the helicity operator reduces to

$$h\epsilon^\mu_\pm = \frac{\vec{p}\cdot\vec{J}}{|\vec{p}|}\epsilon^\mu_\pm = J^3\epsilon^\mu_\pm = \begin{pmatrix}0&0&0&0\\0&0&-\mathrm{i}&0\\0&\mathrm{i}&0&0\\0&0&0&0\end{pmatrix}\epsilon^\mu_\pm = \pm\epsilon^\mu_\pm \tag{3.97}$$

The polarization vectors satisfy the completeness relation

$$\sum_\lambda \epsilon^\mu_\lambda\epsilon^{\nu*}_\lambda = \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} = -g^{\mu\nu} + n^\mu\bar{n}^\nu + \bar{n}^\mu n^\nu \tag{3.98}$$

Together with the light-like vectors

$$\bar{n}^\mu = \frac{1}{\sqrt{2}}(1,0,0,1)\ , \qquad n^\mu = \frac{1}{\sqrt{2}}(1,0,0,-1)\ , \qquad n\cdot\bar{n} = 1 \tag{3.99}$$

the polarization vectors therefore form a complete basis of Minkowski space. Note that we can write

$$\bar{n} = \frac{p^\mu}{(p\cdot n)} \tag{3.100}$$

The completeness relation in the form

$$\sum_\lambda \epsilon^\mu_\lambda \epsilon^{\nu *}_\lambda = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{(p \cdot n)} \tag{3.101}$$

then holds in any reference frame where n can be chosen as a light-like vector orthogonal to the momentum,

$$n^2 = 0 \qquad n \cdot \epsilon_\pm = 0 \tag{3.102}$$

The necessity to describe massless spin one-particles by a vector field with a gauge freedom can be traced back to the impossibility to single out two transverse polarization vectors in a Lorentz invariant way. This can be seen from the fact that Lorentz transformations that leave the momentum invariant (i.e. elements of the little group) do not transform the subspace spanned by the transverse polarizations into itself. To see this consider the infinitesimal transformation with the generators A and B of the little group

$$\Lambda = 1 - \mathrm{i}(\alpha A + \beta B) = \begin{pmatrix} 1 & -\beta & \alpha & 0 \\ -\beta & 1 & 0 & \beta \\ \alpha & 0 & 1 & -\alpha \\ 0 & -\beta & \alpha & 1 \end{pmatrix} \tag{3.103}$$

They satisfy

$$\Lambda \begin{pmatrix} p \\ 0 \\ 0 \\ p \end{pmatrix} = \begin{pmatrix} p \\ 0 \\ 0 \\ p \end{pmatrix}, \qquad \Lambda \begin{pmatrix} 0 \\ \epsilon_1 \\ \epsilon_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \epsilon_1 \\ \epsilon_2 \\ 0 \end{pmatrix} + (\alpha\epsilon_2 - \beta\epsilon_1) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.104}$$

Therefore Lorentz transformations that leave the momentum invariant can change the polarization vectors by a contribution $\sim p^\mu$. Since this "scalar" polarization is not physical, the replacement

$$\epsilon^\mu \to \epsilon^\mu + a p^\mu \tag{3.105}$$

with a arbitrary must not have an effect in physical observables.

3.3 QED

Electromagnetic field coupled to fermion with charge q:

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(\mathrm{i} \not{D} - m \right) \psi \tag{3.106}$$

- Field-strength tensor

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{3.107}$$

Components of four-vector potential: scalar potential and vector potential:

$$A^\mu(x) = \begin{pmatrix} \phi(x) \\ A^i(x) \end{pmatrix} \tag{3.108}$$

The components of the field-strength tensor give the usual expressions for the electric and magnetic fields:

$$\begin{aligned} E^i &= F^{i0} = \partial^i \phi - \partial^0 A^i \\ \Rightarrow \quad \vec{E} &= -(\vec{\nabla}\phi + \frac{1}{c}\dot{\vec{A}}) \\ B^i &= -\frac{1}{2}\epsilon^{ijk}F^{jk} = -\epsilon^{ijk}\partial^j A^k = (\nabla \times A)^i \end{aligned} \tag{3.109}$$

- Relation of field strength to vector potential not unique: "gauge invariance"

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \omega \,, \qquad\qquad F'_{\mu\nu} = F_{\mu\nu} \tag{3.110}$$

- Free relativistic spin 1/2 fermion field ψ: four-component spinor satisfying Dirac equation:

$$(\mathrm{i}\partial\!\!\!/ - m)\psi = 0 \tag{3.111}$$

Conjugate Dirac spinor:

$$\bar{\psi} = \psi^\dagger \gamma^0 \tag{3.112}$$

- covariant derivative

$$D_\mu \;=\; \partial_\mu + \mathrm{i}q A_\mu(x) \tag{3.113}$$

- Lagrangian invariant under gauge transformation (3.110) and

$$\psi(x) \to \psi'(x) = \mathrm{e}^{-\mathrm{i}q\omega(x)}\psi(x) \tag{3.114}$$

since

$$\begin{aligned} D_\mu\psi &\to (\partial_\mu + \mathrm{i}qA'_\mu)\psi' \\ &= ((\partial_\mu + \mathrm{i}q(A_\mu + \partial_\mu\omega))\,\mathrm{e}^{-\mathrm{i}q\omega}\psi \quad = \mathrm{e}^{-\mathrm{i}q\omega} D_\mu\psi \end{aligned} \tag{3.115}$$

- With a view to the later generalization we note that the field-strength tensor can be computed from a commutator of covariant derivatives:

$$[D_\mu, D_\nu]\psi = \mathrm{i}q(\partial_\mu A_\nu - \partial_\mu A_\nu)\psi = \mathrm{i}qF_{\mu\nu}\psi \tag{3.116}$$

- Write Lagrangian as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\,(\mathrm{i}\partial\!\!\!/ - m)\,\psi - A_\mu j^\mu \tag{3.117}$$

with current

$$j^\mu = q\bar{\psi}\gamma^\mu\psi \tag{3.118}$$

Noether theorem for invariance under "global" symmetry transformation (3.114) with $\omega =$ const. implies

$$\partial_\mu j^\mu = 0 \tag{3.119}$$

- "Gauge fixing" term added for quantization. Covariant choice:

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{3.120}$$

  Alternative: axial gauge

$$\mathcal{L}_{\mathrm{gf,ax}} \;=\; -\frac{1}{2\xi}(n^\mu A_\mu)^2, \tag{3.121}$$

  with a constant four-vector n^μ.

- Photon propagator for gauge fixing (3.120) can be derived as Green function to operator

$$\mathcal{D}_x^{\mu\nu} = g^{\mu\nu}\Box_x - \left(1 - \frac{1}{\xi}\right)\partial_x^\mu\partial_x^\nu\,, \tag{3.122}$$

$$\mathcal{D}_x^{\mu\nu} D_{F,\nu\rho}(x,y) = \mathrm{i}g_\rho^\mu \delta^4(x-y) \tag{3.123}$$

  Solution in momentum space:

$$D_F^{\mu\nu}(p^2) = \frac{\mathrm{i}}{p^2 + \mathrm{i}\epsilon}\left(-g^{\mu\nu} + \frac{\mathrm{i}p^\mu p^\nu}{(p^2+\mathrm{i}\epsilon)}(1-\xi)\right) \tag{3.124}$$

- Scalar QED:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi)\\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\phi)^\dagger(\partial^\mu\phi) + \mathrm{i}qA_\mu(\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi) + q^2\phi^\dagger\phi A^2\end{aligned} \tag{3.125}$$

3.4 Feynman rules

3.4.1 S-matrix and Cross section

S-matrix

- Consider scattering process

$$p_1 + p_2 \to k_1 + \ldots k_n \tag{3.126}$$

- Prepared incoming states (Schrödinger representation):

$$|i(t)\rangle \xrightarrow{t \to t_i \to -\infty} \text{free momentum eigenstate } |\phi_{p_1}\phi_{p_2}\rangle \tag{3.127}$$

- Evolve $|i(t)\rangle$ from t_i to t_f:

$$|i(t_f)\rangle = U(t_f, t_i)\,|i(t_i)\rangle \tag{3.128}$$

with time evolution operator $U(t_1, t_2)$.

- Project on states $|f(t)\rangle$:

$$|f(t)\rangle \xrightarrow{t \to t_f \to +\infty} \text{free momentum eigenstates} \tag{3.129}$$

- Definition of **S-matrix element**:

$$S_{fi} = \lim_{t_{i/f} \to \pm\infty} \langle f(t_f)|U(t_f, t_i)|i(t_i)\rangle \tag{3.130}$$

- Transformation to Heisenberg picture at t_0:

$$|\psi, t_0\rangle_H = |\psi(t_0)\rangle = U(t_0, t)\,|\psi(t)\rangle \tag{3.131}$$

$$S_{fi} = {}_H\langle f, t_0|i, t_0\rangle_H \equiv {}_H\langle f, t_f|S|i, t_i\rangle_H \tag{3.132}$$

In the last step the S-matrix element has been expressed in terms of a matrix element of the operator

$$S = \lim_{t_{i/f} \to \pm\infty} U(t_f, t_i) \tag{3.133}$$

between the states $|i, t_i\rangle_H$ and $|f, t_f\rangle_H$, which are approximated by the free momentum eigenstates.

In the following we abbreviate $|i, t_0\rangle_H \to |i\rangle$, $|f, t_0\rangle_H \to |f\rangle$ and $|i, t_i\rangle_H \to |i\rangle_0$, $|f, t_f\rangle_H \to |f\rangle_0$.

Poincaré invariance of the S-matrix

The Poincaré transformations of the initial and final states are obtained by the unitary operators introduced in Section 3.1:

$$|f'\rangle = U(\Lambda, a)\,|f\rangle \ , \qquad |i'\rangle = U(\Lambda, a)\,|i\rangle \tag{3.134}$$

The unitarity of the operators $U(\Lambda, a)$ implies the invariance of the S-matrix elements:

$$S_{f'i'} = \langle U(\Lambda, a)f|U(\Lambda, a)i\rangle = \langle f|U^\dagger(\Lambda, a)U(\Lambda, a)|i\rangle = S_{fi} \tag{3.135}$$

The transformation of momentum eigenstates of massless particles with helicity s under translations (3.50) and Lorentz transformations (3.37) implies:

$$U(1,a)\,|\phi^{s_1}_{p_1}\dots\phi^{s_n}_{p_n}\rangle = e^{-\mathrm{i}a\cdot(p_1+\dots p_n)}\,|\phi^{s_1}_{p_1}\dots\phi^{s_n}_{p_n}\rangle \tag{3.136}$$

$$U(\Lambda,0)\,|\phi^{s_1}_{p_1}\dots\phi^{s_n}_{p_n}\rangle = e^{\mathrm{i}\theta(s_1+\dots s_n)}\,|\phi^{s_1}_{\Lambda p_1}\dots\phi^{s_n}_{\Lambda p_n}\rangle \tag{3.137}$$

The transformation of the states under a translation implies the identity

$$S_{fi} = e^{-\mathrm{i}a(p_1+p_2-k_1-\dots k_n)}S_{fi} \tag{3.138}$$

Therefore, Poincaré invariance implies momentum conservation:

$$p_1+p_2 = k_1+\cdots+k_n. \tag{3.139}$$

For massless states, the transformation of the states under Lorentz-transformations implies

$$S_{fi} = e^{i\theta(s_{i,1}+s_{i,2}-s_{f,1}-\dots s_{f,n})}S_{\bar{f}\bar{i}} \tag{3.140}$$

Here the $S_{\bar{f}\bar{i}}$ is the S-matrix for the states with boosted momenta Λp, but unchanged helicities. For massive states, an analogous result follows from (3.37).

T-matrix and scattering amplitude

T-matrix: extract trivial part of S-matrix

$$S = \mathbb{I}+T \tag{3.141}$$

$${}_0\langle f|S|i\rangle_0 = \underbrace{{}_0\langle f|i\rangle_0}_{\propto\delta_{if}} + {}_0\langle f|T|i\rangle_0 \tag{3.142}$$

Definition of the **scattering amplitude** (transition matrix element) $\mathcal{M}_{fi}$: extract the overall delta function from momentum conservation:

$${}_0\langle f|T|i\rangle_0 \quad = \quad \mathrm{i}(2\pi)^4\,\delta^4\Big(\sum_i k_i^\mu - \sum_j p_j^\mu\Big)\mathcal{M}_{fi} \tag{3.143}$$

Unitarity of the S-matrix

The S matrix maps the asymptotic free states defined at t_i to the corresponding states at t_f:

$$S\,|i,t_i\rangle_H = |i,t_f\rangle_H \tag{3.144}$$

The Unitarity of the S-matrix from completeness/orthogonality of initial/final states:

$$\sum_X \underbrace{\langle i'|X\rangle}_{S^*_{Xi'}}\underbrace{\langle X|i\rangle}_{S_{Xi}} = \langle i'|i\rangle = \delta_{ii'} \tag{3.145}$$

This implies the operator relations:

$$\mathbb{I} = S^\dagger S \quad\Rightarrow\qquad \mathrm{i}T^\dagger T = (T-T^\dagger) \tag{3.146}$$

Implications of unitarity will be discussed in Chapter **??**.

Cross section

Relation of cross section to scattering amplitudes

$$\mathrm{d}\sigma = \underbrace{\frac{1}{4p_1^0 p_2^0 v_{\mathrm{rel}}}}_{\text{flux factor}} |\mathcal{M}_{fi}|^2 \underbrace{\left(\prod_{l=1}^{n} \frac{\mathrm{d}^3 k_l}{(2\pi)^3 (2k_l^0)}\right) (2\pi)^4 \delta(p_1 + p_2 - \sum k_f)}_{= \mathrm{d}\Phi_f,\ \text{invariant phase space volume}} . \tag{3.147}$$

The prefactor can be written in terms of Lorentz invariants

$$p_1^0 p_2^0 v_{\mathrm{rel}} = \sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2} \tag{3.148}$$

For a $2 \to 2$ scattering process the phase-space integral can be simplified to

$$\begin{aligned} \int \mathrm{d}\Phi_2 &= \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2p_1^0} \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2p_2^0} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \Bigg|_{p_{1,2}^0 = \sqrt{m_{1,2}^2 + \vec{p}_{1,2}^2}} \\ &= \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{8s} \theta(\sqrt{s} - m_1 - m_2) \int \mathrm{d}\Omega_1, \end{aligned} \tag{3.149}$$

where $s = (p_1 + p_2)^2$, Ω_1 is the solid angle of particle 1, and

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{3.150}$$

For the special cases of $m_1 = m_2 = m$ and $m = 0$ the phase space simplifies further to

$$\begin{aligned} \int \mathrm{d}\Phi_2 &\overset{m_1=m_2=m}{=} \frac{1}{8(2\pi)^2} \sqrt{1 - \frac{4m^2}{s}}\, \theta(\sqrt{s} - 2m) \int \mathrm{d}\Omega_1, \\ &\overset{m=0}{=} \frac{1}{8(2\pi)^2} \int \mathrm{d}\Omega_1, \end{aligned} \tag{3.151}$$

3.4.2 Perturbation theory and Feynman rules

Consider a Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{3.152}$$

for a set of fields Φ_i (in a symbolic notation with spinor or Lorentz indices suppressed). The **free Lagrangian** can be written as[1].

$$\mathcal{L}_0 = \sum_i \Phi_i(x) \mathcal{D}_i(x) \Phi_i(x) \tag{3.153}$$

with some differential operators $\mathcal{D}_i(x)$. We consider an **interaction Lagrangian** of the form

$$\mathcal{L}_I = g\Gamma_{ijk}^3 \Phi_i \Phi_j \Phi_k + g^2 \Gamma_{ijkl}^4 \Phi_i \Phi_j \Phi_k \Phi_l \tag{3.154}$$

[1]A possible factor 1/2 for real fields is suppressed

where the Γ^n can also contain differential operators. We have extracted a dimensionless coupling constant g. Assuming that g is small enough, a perturbative expansion of the S-matrix can be performed, resulting in the Feynman rules.

For a field Φ we have a mode expansion of the form

$$\Phi(x) = \sum_\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0} \left(\eta_\lambda^-(p) a_\lambda(\vec{p}) e^{-\mathrm{i}p\cdot x} + \eta_\lambda^+(p) b_\lambda^\dagger(\vec{p}) e^{\mathrm{i}p\cdot x} \right) \tag{3.155}$$

where $\eta^\mp$ is the "wave function" (spinor, polarization vector) of particle ϕ /antiparticle $\bar{\phi}$ with helicity λ.

External states: contractions with fields defined as

$$\begin{aligned} \Phi(x)|\phi_p\rangle &= [\Phi(x), a(\vec{p})^\dagger]_\pm = \eta^-(p)\mathrm{e}^{-\mathrm{i}p\cdot x}, & \Phi^\dagger(x)|\bar{\phi}_p\rangle &= \eta^{+*}(p)\mathrm{e}^{-\mathrm{i}p\cdot x}, \\ \langle\phi_p|\Phi^\dagger(x) &= \eta^{-*}(p)\mathrm{e}^{\mathrm{i}p\cdot x}, & \langle\bar{\phi}_p|\Phi(x) &= \eta^+(p)\mathrm{e}^{\mathrm{i}p\cdot x}. \end{aligned} \tag{3.156}$$

$\Rightarrow$ The role of incoming/outgoing wavefunctions is exchanged for particle/antiparticle.

Propagators for the fields Φ_i are defined as Green functions for the kinetic operators $\mathcal{D}$:

$$\mathcal{D}_i(x) D_{F,i}(x,y) = \mathrm{i}\delta^4(x-y) \tag{3.157}$$

with the Fourier transform:

$$D_{F,i}(x,y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-\mathrm{i}p(x-y)} D_{F,i}(p^2) \tag{3.158}$$

where the Feynman $\mathrm{i}\epsilon$ prescription is understood.

Vertex functions V^n with $n = 3, 4$ are defined as

$$\text{(i)} \int \mathrm{d}^4 x \, \langle 0|\mathcal{L}_I(x)|\phi_{p_1}, \ldots, \phi_{p_n}\rangle = (2\pi)^4 \delta^4(\sum_i p_i) V^n_{1\ldots n}(p_1, \ldots, p_n)\eta_1(p_1)\ldots\eta_n(p_n) \tag{3.159}$$

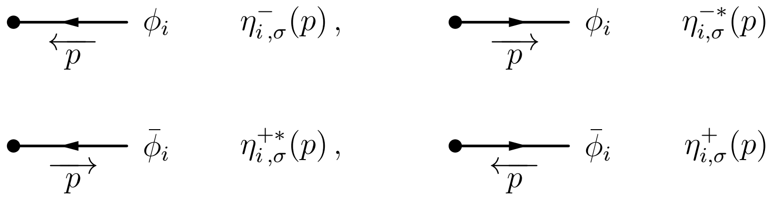
$\Rightarrow$ all possible contractions of fields in $\mathcal{L}_I$ with external states. (exclude contractions within $\mathcal{L}_I$)

Feynman rules

1. Determine all relevant Feynman diagrams:
   - $n \to m$ scattering process $\Rightarrow$ $n + m$ external lines.
   - Order of perturbation theory $\Rightarrow$ number of loops.
2. Impose momentum conservation at each vertex.

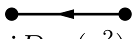
3. Insert the explicit expressions

   - external lines: wave functions η



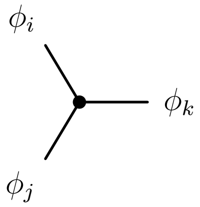
(3.160)

   - internal lines: "propagator":



$$\mathrm{i}D_{F\,i}(p^2) \tag{3.161}$$

   - n-point Vertices from $\Gamma^{(n)}$:



$$V^3_{ijk}(p_1, p_2, p_3) \tag{3.162}$$

4. Insert a relative sign between diagrams that result from interchanging external fermion lines.

5. The coherent sum of all diagrams yields $\mathrm{i}\mathcal{M}_{fi}$.

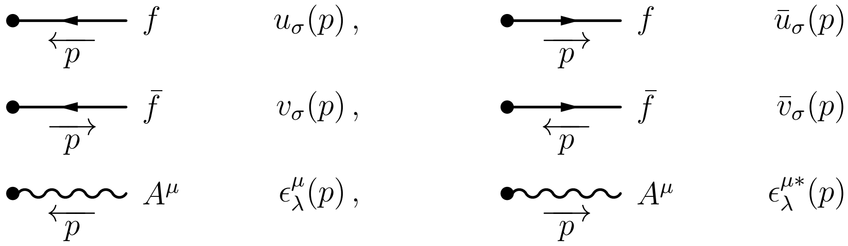
6. Additional rules for diagrams with closed loops:

   - Integrate over all loop momenta p_l via $\displaystyle\int \frac{\mathrm{d}^4 p_l}{(2\pi)^4}$.
   - For each closed fermion loop take Dirac trace and multiply by (-1).

3.4.3 Feynman rules in QED

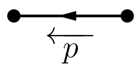
Explicit rules for QED:

External lines:

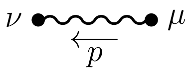


(3.163)

Propagators:

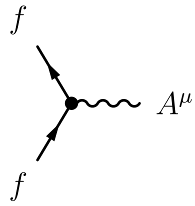


$$\frac{\mathrm{i}}{\not{p} - m_f + \mathrm{i}\epsilon},$$



$$\frac{-\mathrm{i}g^{\mu\nu}}{p^2 + \mathrm{i}\epsilon} + \frac{\mathrm{i}p^\mu p^\nu}{(p^2 + \mathrm{i}\epsilon)^2}(1 - \xi) \tag{3.164}$$

Vertex



$$-\mathrm{i}Q_f e\gamma^\mu \tag{3.165}$$

3.4.4 Feynman rules for interactions with momenta

Example: scalar QED

$$\mathcal{L}_I^{(3)} = \Gamma^{3,\mu}_{A\phi^\dagger\phi} A_\mu \phi^\dagger \phi = \mathrm{i}eA^\mu(\phi^\dagger \partial_\mu \phi - (\partial_\mu \phi^\dagger)\phi) \tag{3.166}$$

The contractions with derivatives of field operators result in factors of the momenta of the external states:

$$\partial_\mu \Phi(x)|\phi_p\rangle = \eta^-(p)(-\mathrm{i}p_\mu)\mathrm{e}^{-\mathrm{i}px}, \qquad \langle\phi_p|\partial_\mu \Phi^\dagger(x) = \eta^{-*}(p)(\mathrm{i}p^\mu)\mathrm{e}^{\mathrm{i}px}. \tag{3.167}$$

One possible contraction, e.g.:

$$A^\mu(x)\phi^\dagger(x)\partial_\mu\phi(x)|\gamma_\lambda^{p_1}\phi^{p_2}\bar{\phi}^{p_3}\rangle = e^{-\mathrm{i}(p_1+p_2+p_3)\cdot x}\epsilon_\lambda(p_1)(-\mathrm{i}p_{2,\mu}) \tag{3.168}$$

where $|\phi\rangle$ is the scalar particle state annihilated by ϕ and $|\bar{\phi}\rangle$ the anti-particle state annihilated by $\phi^\dagger$.

The Feynman rule of the vertex gives

$$\begin{aligned} \mathrm{i}e \int \mathrm{d}^4x\, \langle 0|A^\mu(\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi)|\gamma_\lambda^{p_1}\phi_-^{p_2}\phi_+^{p_3}\rangle \\ = \mathrm{i}e \int \mathrm{d}^4x e^{-\mathrm{i}(p_1+p_2+p_3)\cdot x}(-\mathrm{i})(p_{2,\mu} - p_{3,\mu})\epsilon_\lambda^\mu(p_1) \\ = \mathrm{i}(2\pi)^4\delta(p_1+p_2+p_3)\,(-\mathrm{i})e(p_{2,\mu} - p_{3,\mu})\epsilon^\mu(p_1) \end{aligned} \tag{3.169}$$

i.e. the vertex function is

$$V^{3,\mu}_{\gamma\phi^-\phi^+}(p_1,p_2,p_3) = (-\mathrm{i})e(p_2^\mu - p_3^\mu) \tag{3.170}$$

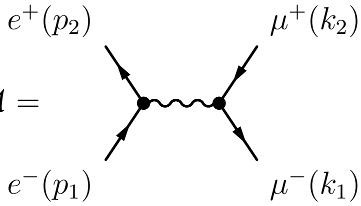
3.5 Calculation of cross sections

3.5.1 Example: $e^-e^+ \to \mu^-\mu^+$

Textbook approach for the calculation of spin-averaged cross sections for the example of $e^-(p_1)e^+(p_2) \to \mu^-(k_1)\mu^+(k_2)$

- Compute matrix element $\mathrm{i}\mathcal{M}$ from Feynman rules:

$\mathrm{i}\mathcal{M} =$



$$= (-\mathrm{i}e)^2(\bar{u}_{\sigma_1}(k_1)\gamma_\mu v_{\sigma_2}(k_2))\frac{-\mathrm{i}}{p^2+\mathrm{i}\epsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2+\mathrm{i}\epsilon}(1-\xi)\right)(\bar{v}_{\lambda_2}(p_2)\gamma_\nu u_{\lambda_1}(p_1)) \tag{3.171}$$

with $p = k_1 + k_2 = p_1 + p_2$. The ξ-dependent part of the propagator drops out using the Dirac equation in momentum space (3.62):[2]

$$p^\nu(\bar{v}(p_2)\gamma_\nu u(p_1)) = \bar{v}(p_2)(\not{p}_1 + \not{p}_2)u(p_1) = \bar{v}(p_2)(m-m)u(p_1) = 0 \tag{3.172}$$

It can be shown that this feature generalizes to all amplitudes in QED and that the terms $\sim p^\mu p^\nu$ in the photon propagator always drops out. Therefore one can chose $\xi = 1$ ("Feynman gauge").

- To calculate an unpolarized cross section, the squared matrix element is averaged over initial-state spins and summed over final-state spins:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &\equiv \frac{1}{2}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&= \frac{e^4}{4s^2}\sum_{\lambda_i,\sigma_i}(\bar{u}_{\sigma_1}(k_1)\gamma_\mu v_{\sigma_2}(k_2))\underbrace{(\bar{u}_{\sigma_1}(k_1)\gamma_\nu v_{\sigma_2}(k_2))^*}_{=\bar{v}_{\sigma_2}(k_2)\gamma^0\gamma_\nu^\dagger\gamma^0 u_{\sigma_1}(k_1)} \\
&\qquad\times(\bar{v}_{\lambda_2}(p_2)\gamma^\mu u_{\lambda_1}(p_1))(\bar{v}_{\lambda_2}(p_2)\gamma^\nu u_{\lambda_1}(p_1))^* \\
&= \frac{e^4}{4s^2}\sum_{\lambda_i,\sigma_i}(\bar{u}_{\sigma_1}(k_1)\gamma_\mu v_{\sigma_2}(k_2))(\bar{v}_{\sigma_2}(k_2)\gamma_\nu u_{\sigma_1}(k_1)) \\
&\qquad\times(\bar{v}_{\lambda_2}(p_2)\gamma^\mu u_{\lambda_1}(p_1))(\bar{u}_{\lambda_1}(p_1)\gamma^\nu v_{\lambda_2}(p_2))
\end{aligned} \tag{3.173}$$

where we have used

$$\bar{u}^\dagger = \gamma^0 u\,, \tag{3.174}$$

$$\gamma^{0\dagger} = \gamma^0\,, \qquad \gamma^{i\dagger} = -\gamma^i \qquad \Rightarrow \qquad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{3.175}$$

[2] Note that this only works since the vertex involves fermions with the same mass (there is no muon-electron photon vertex in QED!).

- The spin sums can be performed using the completeness relations of the Dirac spinors

$$\sum_{\lambda=1,2} u_{\lambda a}(p)\bar{u}_{\lambda b}(p) = (\not{p}+m)_{ab}, \qquad \sum_{\lambda=1,2} v_{\lambda a}(p)\bar{v}_{\lambda b}(p) = (\not{p}-m)_{ab} \tag{3.176}$$

where the spinor indices have been made explicit. As a result one obtains the expression

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2}\,\mathrm{tr}[((\not{k}_1+m_\mu)\gamma_\mu(\not{k}_2-m_\mu)\gamma_\nu]\,\mathrm{tr}[(\not{p}_2-m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu] \tag{3.177}$$

- The resulting traces over gamma matrices can be evaluated using identities derived from the Dirac algebra such as

$$\mathrm{tr}[\gamma^\mu\gamma^\nu] = \frac{1}{2}\,\mathrm{tr}[\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu] = g^{\mu\nu}\,\mathrm{tr}\,\mathbf{1} = 4g^{\mu\nu} \tag{3.178a}$$

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\sigma] = 0 \tag{3.178b}$$

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\rho}g^{\nu\sigma}) \tag{3.178c}$$

This gives for example

$$\mathrm{tr}[(\not{p}_2-m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu] = 4\left(p_2^\mu p_1^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}\left[(p_1\cdot p_2)+m_e^2\right]\right) \tag{3.179}$$

The spin-averaged matrix element gives

$$\overline{|\mathcal{M}|^2} = \frac{4e^4}{s^2}2\left[(p_1\cdot k_1)(p_2\cdot k_2)+(p_1\cdot k_2)(k_1\cdot p_2)+(p_1\cdot p_2)m_\mu^2+(k_1\cdot k_2)m_e^2+2m_\mu^2 m_e^2\right] \tag{3.180}$$

- Cross section for $m_e \to 0$: use cms frame:

$$\begin{aligned} p_{1,2}^\mu &= (p,0,0,\pm p), \\ k_1^\mu &= (E,\pm k\cos\phi\sin\theta,\pm k\sin\phi\sin\theta,\pm k\cos\theta). \end{aligned} \tag{3.181}$$

with $E=\sqrt{k^2+m_\mu^2}$. The squared matrix element becomes ($s=4p^2=4E^2$)

$$\overline{|\mathcal{M}|^2} = 2e^4\left[1+\frac{4m_\mu^2}{s}+\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta\right] \tag{3.182}$$

- The differential cross section if found by inserting the squared matrix element into (3.147) using the phase-space integral (3.149)

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\phi\mathrm{d}\cos\theta} &= \underbrace{\frac{1}{4p_1^0p_2^0v_{\mathrm{rel}}}}_{8p^2=2s}\;\frac{1}{8(2\pi)^2}\sqrt{1-\frac{4m_\mu^2}{s}}\overline{|\mathcal{M}|^2} \\ &= \frac{\alpha^2}{4s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[1+\frac{4m_\mu^2}{s}+\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta\right] \end{aligned} \tag{3.183}$$

The total cross section is obtained by integrating over the angles:

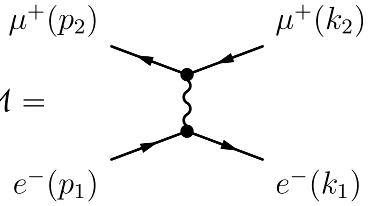
$$\begin{aligned}\sigma &= \frac{\pi\alpha^2}{s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[1+\frac{4m_\mu^2}{s}+\frac{1}{3}\left(1-\frac{4m_\mu^2}{s}\right)\right] \\ &= \frac{4\pi\alpha^2}{3s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[1+\frac{2m_\mu^2}{s}\right]\end{aligned} \tag{3.184}$$

3.5.2 Remarks on the calculation of scattering amplitudes

Crossing: $e^-\mu \to e^-\mu$

- Matrix element for $e^-(p_1)\mu^+(p_2) \to e^-(k_1)\mu^+(k_2)$

$$\mathrm{i}\mathcal{M} =$$



$$= (-\mathrm{i}e)^2(\bar{v}_{\sigma_2}(p_2)\gamma_\mu v_{\lambda_2}(k_2))\frac{-\mathrm{i}g^{\mu\nu}}{(p_1-k_1)^2+\mathrm{i}\epsilon}(\bar{u}_{\sigma_1}(k_1)\gamma_\nu u_{\lambda_1}(p_1)) \tag{3.185}$$

Comparison to the matrix element for $e^-(p_1)e^+(p_2) \to \mu^-(k_1)\mu^+(k_2)$ (3.171): change incoming $e^+(p_2)$ to outgoing $e^-(k_1)$:

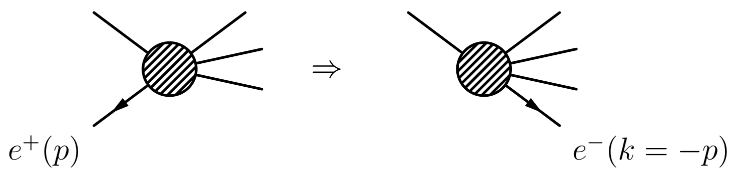
$$\begin{aligned}\bar{v}(p_2) &\to \bar{u}(k_1) \\ \bar{u}(k_1) &\to \bar{v}(p_2) \\ (p_1+p_2)^2 &\to (p_1-k_1)^2\end{aligned} \tag{3.186}$$

- Spin averaged matrix element

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{e^4}{4t^2}\,\mathrm{tr}[((\not{p}_2-m_\mu)\gamma_\mu(\not{k}_2-m_\mu)\gamma_\nu]\,\mathrm{tr}[(\not{k}_1+m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu] \\ &= \frac{8e^4}{t^2}\left[(k_1\cdot k_2)(p_1\cdot p_2)+(p_1\cdot k_2)(p_2\cdot k_1)-(p_2\cdot k_2)m_e^2-(p_1\cdot k_1)m_\mu^2+2m_\mu^2m_e^2\right]\end{aligned} \tag{3.187}$$

- Relation to squared matrix element for $e^-e^+ \to \mu^-\mu^+$:
  (3.173): **crossing of momenta**

$$p_2 \leftrightarrow -k_1 \tag{3.188}$$



(3.189)

Reason: Completeness relation for (anti-)muon spinors

$$\sum_\sigma v_\sigma(p_2)\bar{v}_\sigma(p_2) = (\not{p}_2 - m_\mu) \to -(\not{k}_1 + m_\mu) = -\sum_\sigma u_\sigma(k_1)\bar{u}_\sigma(k_1) \tag{3.190}$$

External photons and gauge invariance

Gauge invariance implies that scattering amplitudes must be invariant under the replacement 3.105

$$\epsilon^\mu \to \epsilon^\mu + ap^\mu \tag{3.191}$$

of the polarization vectors. This implies that amplitudes $\mathcal{M}$ with an incoming/outgoing photon of momentum p obey the following **Ward identity**:

$$\mathcal{M} \;=\; \epsilon^\mu_\lambda(p)^{(*)}\,\tilde{\mathcal{M}}_\mu(p) \quad \Rightarrow \quad p^\mu\,\tilde{\mathcal{M}}_\mu(p) \;=\; 0, \tag{3.192}$$

where all external states other than the photon must be on shell. In the above example, we can use the Dirac equations (3.62) for the spinors

$$(\not{p}_1 - m_e)u(p_1) = 0, \qquad \bar{v}(p_2)(\not{p}_2 + m_e) = 0 \tag{3.193}$$

to check that the Ward identity is satisfied:

$$\begin{aligned}
\mathrm{i}\tilde{\mathcal{M}}_\mu k_1^\mu &= (-\mathrm{i}e)^2\bar{v}(p_2)\left[\not{\epsilon}^*(k_2)\frac{\mathrm{i}}{\not{p}_1 - \not{k}_1 - m_e}\not{k}_1 + \not{k}_1\frac{\mathrm{i}}{\not{p}_1 - \not{k}_2 - m_e}\not{\epsilon}^*(k_2)\right]u(p_1) \\
&= (-\mathrm{i}e)^2\bar{v}(p_2)\Bigg[\not{\epsilon}^*(k_2)\frac{\mathrm{i}}{\not{p}_1 - \not{k}_1 - m_e}(\not{k}_1 - (\not{p}_1 - m_e)) \\
&\qquad + (\not{k}_1 - (\not{p}_2 + m_e))\frac{\mathrm{i}}{\not{k}_1 - \not{p}_2 - m_e}\not{\epsilon}^*(k_2)\Bigg]u(p_1) \\
&= (-\mathrm{i}e)^2\bar{v}(p_2)\mathrm{i}\left[-\not{\epsilon}^*(k_2) + \not{\epsilon}^*(k_2)\right]u(p_1) = 0
\end{aligned} \tag{3.194}$$

- In general only the sum of all diagrams contributing to an amplitude satisfies the Ward identity, not individual diagrams.
- A general diagrammatic proof of the Ward identity can be found e.g. in [3].
- In QED the Ward identity allows to simplify the calculation of spin summed cross sections by dropping the momentum dependent terms in the polarization sum (3.101):

$$\sum_\lambda \epsilon^\mu_\lambda \epsilon^{\nu *}_\lambda = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{(p \cdot n)} \Rightarrow -g^{\nu\nu} \tag{3.195}$$

  For processes with more than one external photon, this is only possible if the identity (3.192) continues to hold if all external photon polarization vectors are stripped off the amplitude. This holds in general in QED, but not in nonabelian gauge theories such as QCD.

Remarks on calculations for many-particle final states

- The number of Feynman diagrams increases drastically with the number of external particles.
- For N contributing Feynman diagrams to $\mathcal{M}$, there are N^2 terms in the squared matrix element $|\mathcal{M}|^2$. The above procedure therefore can become very lengthy already for a moderate number of diagrams.
- Computer programs can be used for the computation of Dirac-traces, e.g. tracer, Form, FeynCalc, FormCalc
- In numerical calculations: implement explicit expressions for spinors and polarization vectors and compute **helicity amplitudes**

$$\mathcal{M}(\lambda_1, \lambda_2, \dots) \tag{3.196}$$

  For each phase-space points this evaluates to a complex number, so the computation of $|\mathcal{M}|^2$ is simple. The sum over helicities is performed numerically. In this approach one needs to evaluate N Feynman diagrams 2^n times for n external particles.

- For massless particles, the vector interaction is helicity conserving:

$$\bar{u}_{\lambda'}\gamma^\mu u_\lambda = \begin{cases} (u_+^\dagger \bar{\sigma}^\mu u_+)\,, & (\lambda, \lambda') = (R, R) \\ (u_-^\dagger \sigma^\mu u_-)\,, & (\lambda, \lambda') = (L, L) \\ 0\,, & (\lambda, \lambda') = (R, L)\,(L, R) \end{cases} \tag{3.197}$$

  $\Rightarrow$ fewer non-vanishing helicity combinations, amplitudes simplify in terms of two-component spinors.

Chapter 4

Introduction to QCD

As discussed in Chapter 2, quarks carry an additional quantum number "colour". Therefore the quark field is described by three complex Dirac spinors:

$$Q_i(x)\,, \quad i = 1, 2, 3 \tag{4.1}$$

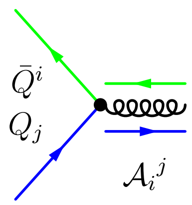
The theory should be invariant under rotations of the quarks in colour space:

$$Q_i \to U_i{}^j Q_j \tag{4.2}$$

where the matrices $U \in SU(3)$ are complex three-by three matrices with unit determinant.

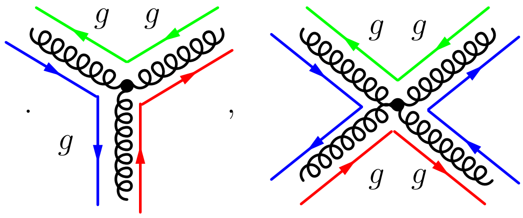
The search for a theory compatible with the confinement of quarks into hadrons at low energies, and with quasi-free quarks at high energies needed to explain deep-inelastic scattering led to the proposal of **vector bosons** ("gluons") as mediators of the interactions among quarks. Gluons are analogous to photons but carry colour quantum numbers (otherwise QCD would just be a copy of QED which cannot explain asymptotic freedom). The theory therefore should include an interaction term of the form

$$\mathcal{L}_\mathrm{I} \sim g_s \bar{Q}^i \gamma^\mu \mathcal{A}_{\mu,i}{}^j Q_j \Rightarrow \tag{4.3}$$



- g_s: "strong coupling constant": analogous to e in QED
- $\mathcal{A}_{\mu,i}{}^j$ gluon field, matrix in colour space so it can change the colour charge of the quarks.
- The interaction Lagrangian is hermitian provided the vector field is hermitian: $\mathcal{A}_{\mu,j}{}^{i*} = \mathcal{A}_{\mu,j}{}^i$.
- Interactions of massless vector bosons must be gauge invariant, $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \alpha$ (for free asymptotic states) because of Lorentz transformations of polarization vectors (3.104).

- Gluons carry colour degree of freedom, therefore self-interactions are expected to be possible:



(4.4)

- The guiding principle in the construction of QCD will be the extension of the $U(1)$ gauge invariance of QCD to $SU(3)$ by allowing the colour transformations of the quarks to depend on space-time:

$$Q_i(x) \to U_i^{\ j}(x) Q_j(x) \tag{4.5}$$

  Such transformations are called **local** $SU(3)$ **transformations**.

- The resulting theory is the unique theory consistent with the Ward-identity (3.192) of scattering amplitudes. As an alternative sketched in [6], the same theory could also be obtained starting with an Ansatz for the interaction vertices (4.3) and (4.3) and imposing the Ward identity.

4.1 $SU(3)$

4.1.1 Generators and Lie Algebra

Lie Groups and generators

The group $SU(3)$ is an example for a Lie Group. We keep the discussion of Lie groups here brief, for more detailed definitions see Chapter 7 of [2].

- $SU(3)$ transformations can be defined by eight real parameters ω^a, $a = 1, \ldots 8$, similar to the three Euler angles parametrizing rotations.
- A group whose elements depend on a set of continuous parameters is called a **Lie group**.
- The number of parameters (e.g. 3 for $SO(3)$, 8 for $SU(3)$) is called the **dimension** of the Lie group.

It is familiar from quantum mechanics that a unitary matrix U can be written in terms of an hermitian matrix H:

$$U = \exp(-\mathrm{i}H), \tag{4.6}$$

with $H = H^\dagger$. On can decompose the hermitian matrix H in a basis of so-called **generators** T^a so that

$$U(\omega) = \exp(-\mathrm{i}\omega^a T^a) \tag{4.7}$$

The condition $\det(U) = 1$ implies that the generators are traceless,

$$\operatorname{tr} T^a = 0 \tag{4.8}$$

since

$$\det \exp A = \exp \operatorname{tr} A \tag{4.9}$$

Gell-Mann matrices

A concrete realization of the traceless, hermitian Generators T^a of $SU(3)$ is given in terms of the **Gell-Mann**-matrices, which are three-dimensional generalizations of the Pauli matrices:

$$T^a = \frac{\lambda^a}{2} \tag{4.10}$$

with

$$\begin{aligned}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{4.11}$$

Lie Algebras

It can be shown that the group-theory axioms imply that the set of generators is closed under forming the commutator, i.e. the commutator of two generators is a linear combination of generators:

$$[T^a, T^b] = \mathrm{i} f^{abc} T^a. \tag{4.12}$$

- A set of objects with this property is called a **Lie algebra**.
- The quantities f^{abc} are called **structure constants** of the Lie algebra.
- From the definition of the structure constants it is clear that $f^{abc} = -f^{bac}$.
- From the **Jacobi Identity** of commutators

  $$[T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0 \tag{4.13}$$

  one obtains an identity of the structure constants (also called Jacobi identity)

  $$f^{bcd} f^{ade} + f^{abd} f^{cde} + f^{cad} f^{bde} = 0. \tag{4.14}$$

- In case the object $g^{ab} = \mathrm{tr}(T^a T^b)$ has only positive eigenvalues, one can choose a basis of generators so that the generators are normalized as

$$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta_{ab}. \tag{4.15}$$

In this basis the structure constants can be obtained from the generators by the relation¿

$$f^{abc} = -2\mathrm{i}\,\mathrm{tr}\left([T^a, T^b]T^c\right) \tag{4.16}$$

In such a basis the structure constants are totally antisymmetric:

$$f^{abc} = -f^{bac} = -f^{acb} = f^{cab} = -f^{cba} = f^{bca}. \tag{4.17}$$

since

$$f^{abc} = -2\mathrm{i}\,\mathrm{tr}\left((T^a T^b - T^b T^a)T^c\right) = -2\mathrm{i}\,\mathrm{tr}\left(T^b T^c T^a - T^c T^b T^a\right) = f^{bca}, \tag{4.18}$$

where the cyclic symmetry of the trace has been used.

4.1.2 Representations

The elements of a group can act on different vector spaces. Familiar examples are the action of Lorentz transformations on tensors of different rank in relativity, or the action of rotations on the spaces of states with different angular momentum in quantum mechanics. The action of a group on a vector space is called a **representation** of the group. The dimension n of the vector space is used to denote the dimensionality of the representation.

Formally, a representation R of a group G on a vector space V is a mapping of elements $g \in G$ to linear transformations $U^{(R)}(g)$ on V that is compatible with the group multiplication,

$$g \cdot f = h \quad \Rightarrow \quad U^{(R)}(g)U^{(R)}(f) = U^{(R)}(g \cdot f) = U^{(R)}(h). \tag{4.19}$$

The vector space V is called "representation space". In physics the elements of the n-dimensional vector space V are often called 'multiplets' (n-plets) and, with an abuse of notation, the vector space itself is often called 'the n-representation'.

Two representations R and R' on a vector space V are called *equivalent*, $R \sim R'$, if there exists an invertible transformation S so that

$$U^{(R')}(g) = SU^{(R)}(g)S^{-1}\ , \quad \forall g \in G.$$

Fundamental representations

In special relativity, tensors can be constructed from co-and contravariant vectors, while in Quantum mechanics states with arbitrary angular momentum can be constructed from spin one-half states. Such representations serving as building blocks for all representations are called **fundamental representations**.

In addition to the quarks with the transformation (4.2), we have anti-quarks that transform with the complex conjugate transformation. For $SU(3)$ the fundamental representations are the 3-dimensional representation appearing in the transformation law of the quarks, and the complex conjugate $\bar{3}$ representation of the antiquark:

$$3: \qquad Q_i \to U_i^{(3)\,j} Q_j = U_i{}^j Q_j \tag{4.20}$$

$$\bar{3}: \qquad \bar{Q}^i \to (U^{(\bar{3})})^i{}_j \bar{Q}^j = \bar{Q}^j (U^\dagger)_j{}^i \tag{4.21}$$

Combining representations

As in relativity, one can consider "tensors" with multiple upper and lower indices. These are obtained for instance by multiplying two quark fields,

$$D_{ij} = Q_i Q_j \tag{4.22}$$

This defines the representation $3 \otimes 3$ with the transformation law

$$(U^{(3\otimes 3)})_{ij}{}^{i'j'} D_{i'j'} = U^{i'}{}_i U^{j'}{}_j D_{i'j'} \tag{4.23}$$

Tensors with several indices can be decomposed into symmetric and antisymmetric contributions:

$$D_{ij} = S_{ij} + A_{ij} \tag{4.24}$$

with

$$S_{ij} = \frac{1}{2}(D_{ij} + D_{ji}) \qquad A_{ij} = \frac{1}{2}(D_{ij} - D_{ji}) \tag{4.25}$$

Note that the symmetry properties are not changed by the colour rotations. Therefore the set of symmetric tensors forms a separate representation of $SU(3)$. Since symmetric 3×3 matrices have six independent entries, it is called the 6 representation.

Out of the antisymmetric part we can form the three-component object

$$A^k = \epsilon^{ijk} A_{ij} \tag{4.26}$$

Because of the identity

$$\epsilon^{i'j'k'} U_{i'}{}^i U_{j'}{}^j = \epsilon^{ijk} (U^\dagger)^{k'}{}_k \tag{4.27}$$

it transforms in the conjugate representation $\bar{3}$:

$$A^{k'} = A^k (U^\dagger)^{k'}{}_k \tag{4.28}$$

(this is analogous to the proof that the vector product $\vec{x} \times \vec{y}$ transforms as a vector under rotations).

This shows that the tensor product of two 3-representation can be decomposed as

$$3 \otimes 3 = \bar{3} \oplus 6 \tag{4.29}$$

Note that for an $SU(N)$ group one can convert $N-1$ lower indices to one upper index using the antisymmetric tensor with N indices. In $SU(2)$ the $\bar{2}$ is equivalent to the 2 so that we do not treat them as independent representations.

The second example is a tensor with one upper and one lower index, obtained for instance by multiplying a quark and an antiquark field:

$$M^i{}_j = \bar{Q}^i Q_j \tag{4.30}$$

These tensors transform in the representation $3 \otimes \bar{3}$:

$$(U^{(3\otimes\bar{3})})_{ji'}{}^{ij'} M^{i'}{}_{j'} = U^i{}_{i'} M^{i'}{}_{j'} (U^\dagger)^{j'}{}_j \tag{4.31}$$

Since the two indices behave differently with respect to $SU(3)$ transformations, it makes no sense to symmetrize or antisymmetrize them. Instead, they can be contracted using the Kronecker delta:

$$M^i{}_j = O^i{}_j + S\delta^i{}_j \tag{4.32}$$

with the "trace part"

$$S = \frac{1}{3} M^i{}_i \tag{4.33}$$

that is invariant under $SU(3)$ transformations. The remaining traceless tensor O has eight independent entries and defines the 8-representation. Therefore we have the decomposition

$$3 \otimes \bar{3} = 1 \oplus 8 \tag{4.34}$$

A representation that cannot be decomposed further is called an **irreducible** representation. Generalizing the above discussion one finds that the irreducible representations in $SU(3)$ are given by tensors with n symmetrized upper and m symmetrized lower indices that are traceless with respect to contractions of upper and lower indices. These concepts are important for the application of $SU(3)$ to the classification of the hadron spectrum where larger representations appear. In this lecture we will only encounter the singlet, triplet and octet representations.

For instance one obtains

$$3 \times 3 \times 3 = (\bar{3} \oplus 6) \times 3 = 1 \oplus 8 \oplus 8 \oplus 10 \tag{4.35}$$

where the 10 is given by symmetric tensors with three lower indices. The decomposition $3 \times 6 = 8 \oplus 10$ can be seen writing the tensor product $Q_i S_{jk}$ with the symmetric tensor $S_{ij} = S_{ji}$ in the 6 representation as

$$Q_i S_{jk} = \frac{1}{3}(Q_i S_{jk} + Q_j S_{ik} + Q_k S_{ij}) + \frac{1}{3}\underbrace{(2Q_i S_{jk} - Q_j S_{ik} - Q_k S_{ij})}_{\epsilon_{ijl}A^l{}_k + \epsilon_{ikl}A^l{}_j} \tag{4.36}$$

where the 8 representation is identified as

$$A^l{}_k = \epsilon^{lmn} Q_m S_{nk} \tag{4.37}$$

This is traceless because of the symmetry of S and the antisymmetry of the epsilon tensor:

$$A^i{}_i = \epsilon^{imn} Q_m S_{ni} = \frac{1}{2}\epsilon^{imn} Q_m (S_{ni} - S_{in}) = 0 \tag{4.38}$$

Representations of Lie algebras

A set of dim(G) matrices $\mathbf{T}^{(R)a}$ that satisfies the same commutator relation (4.39) forms a representation of the Lie algebra:

$$[\mathbf{T}^{(R)a}, \mathbf{T}^{(R)b}] = \mathrm{i} f^{abc} \mathbf{T}^{(R)a}. \tag{4.39}$$

The structure constants do not depend on the representation, but on the basis chosen for the Lie algebra. Generalizing the normalization (4.15), one defines the **index of the representation** T_R:

$$\mathrm{tr}(\mathbf{T}^{(R)a}\mathbf{T}^{(R)b}) = T_R \delta_{ab}. \tag{4.40}$$

For the fundamental representation, the generators are just represented by the Gell-Mann matrices

$$\mathbf{T}^{(3)a} = T^a \tag{4.41}$$

and $T_3 \equiv T_F = \frac{1}{2}$. The generators $\mathbf{T}^{(R)}$ are related to the group elements in the representation R by the usual exponentiation

$$U^{(R)}(\omega) = e^{-\mathrm{i}\omega^a \mathbf{T}^{(R)a}} \tag{4.42}$$

Conjugate representation

For every representation R the conjugate representation $\bar{R}$ with the generators

$$\mathbf{T}^{(\bar{R})a} = -\mathbf{T}^{(R)a,T} = -\mathbf{T}^{(R)*} \tag{4.43}$$

is also a representation. In the last step it was used that the generators are hermitian. For the anti-fundamental representation, this definition is consistent with the transformation (4.21)

$$(U^{(\bar{3})})^i{}_j = (e^{-\mathrm{i}\omega^a \mathbf{T}^{(\bar{3})a}})^i{}_j = (e^{\mathrm{i}\omega^a T^{a,T}})^i{}_j = (U^{(3)\dagger})_j{}^i \tag{4.44}$$

Adjoint representation

For every Lie Algebra there always exists the so called **adjoint** representation given by the structure constants:

$$(T^{(\mathrm{ad})a})_{bc} := -\mathrm{i} f^{abc} \tag{4.45}$$

They satisfy the commutator relation (4.39) due to the Jacobi identity (4.14).

In $SU(N)$ groups the dimension of the adjoint representation is N^2-1, which is identical to the dimension of the traceless tensor representation $M^i{}_j$ in the tensor product of the fund mental and the anti-fundamental representations, $N \otimes \bar{N} = 1 \oplus (N^2 - 1)$. In $SU(3)$, the adjoint representation therefore is identified with the 8-dimensional representation in (4.34).

The index of the adjoint representation of $SU(N)$ is given by (⇒ homework)

$$T_A = N \tag{4.46}$$

Generators of product representations

The generator for a product representation $R \otimes R'$ is given by

$$\mathbf{T}^{(R\otimes R')} = \mathbf{T}^{(R)} \otimes \mathbf{1}^{(R')} + \mathbf{1}^{(R)} \otimes \mathbf{T}^{(R')} \tag{4.47}$$

This follows from expanding the representation of the group elements

$$U^{(R\otimes R')}(\omega) = U^{(R)}(\omega) \otimes U^{(R')}(\omega) = (\mathbf{1}^{(R)} - \mathrm{i}\omega^a \mathbf{T}^{(R)a} + \dots) \otimes (\mathbf{1}^{(R')} - \mathrm{i}\omega^a \mathbf{T}^{(R')a} + \dots) \tag{4.48}$$

to linear order. The most familiar application is the expression of the total angular momentum operator for a two-particle system:

$$\vec{J} = \vec{L}_1 \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes \vec{L}_2 \tag{4.49}$$

4.2 QCD as non-abelian gauge theory

4.2.1 Non-abelian gauge invariance

In order to generalize the notion of gauge invariance, we generalize the gauge transformation of fermions in QED (3.114)

$$\psi(x) \to \psi'(x) = \mathrm{e}^{-\mathrm{i}q\omega(x)} \psi(x) \tag{4.50}$$

by allowing the matrix-valued rotation in colour space to depend on the space-time coordinate:

$$Q_i(x) \to Q_i'(x) = U_i^{\ j}(x) Q_j(x) = (\mathrm{e}^{-\mathrm{i}\omega^a(x) T^a})_i^{\ j} Q_j(x) \tag{4.51}$$

Transformations of this form with space-time dependent elements of a non-abelian group are called **non-abelian gauge transformations**.

Covariant derivative

In the next step, we generalize the covariant derivative $D_\mu = \partial_\mu + \mathrm{i}qA_\mu(x)$ from QED to the colour rotations by postulating a matrix-valued covariant derivative with the property

$$D_{\mu,i}{}^{j} Q_j'(x) = U_i^{\ j}(x) D_{\mu,j}{}^{k} Q_k(x) \tag{4.52}$$

Introducing the gluon field in analogy to the photon field in QED by writing the covariant derivative as

$$D_{\mu i}{}^{j} = \partial_\mu \delta_i{}^j + \mathrm{i} g_s \mathcal{A}_{\mu,i}{}^j(x) \tag{4.53}$$

the condition on the covariant derivative becomes (suppressing the colour indices)

$$D'_\mu(UQ) = \left[\partial_\mu(UQ) + \mathrm{i} g_s \mathcal{A}'_\mu UQ\right] \stackrel{!}{=} U(\partial_\mu + \mathrm{i} g_s \mathcal{A}_\mu)Q \tag{4.54}$$

The derivatives of Q cancel on both sides so one can solve for $\mathcal{A}'_\mu$:

$$\mathcal{A}'_\mu = U\mathcal{A}_\mu U^\dagger + \frac{\mathrm{i}}{g_s}(\partial_\mu U)U^\dagger \tag{4.55}$$

This is the non-abelian generalization of the QED gauge transformation $A'_\mu = A_\mu + \partial_\mu \omega$. For the case $U(x) = \exp(-\mathrm{i}q\omega(x))$ the simpler QED transformation is reproduced.

Field-strength tensor

In order to obtain the non-abelian generalization of the field strength we use the definition

$$[D_\mu, D_\nu]_i{}^j Q_j = \mathrm{i} g_s \mathcal{F}_{\mu\nu,i}{}^j Q_j \tag{4.56}$$

By definition of the covariant derivatives, the so-defined field strength tensor, which is also a matrix in colour space, transforms under gauge transformations as

$$\mathcal{F}'_{\mu\nu} = U\mathcal{F}_{\mu\nu}U^\dagger \tag{4.57}$$

In contrast to the QED case, the field strength tensor in a non-abelian gauge theory is not gauge invariant but transform "covariantly" i.e. like the tensor produce $3 \otimes \bar{3}$ with respect to colour rotations. To calculate the field-strength tensor explicitly, note that the gluon field matrices do not commute with each other. Therefore

$$\mathcal{F}_{\mu\nu,i}{}^j = \partial_\mu \mathcal{A}_{\nu,i}{}^j - \partial_\nu \mathcal{A}_{\mu,i}{}^j + \mathrm{i} g_s [\mathcal{A}_\mu, \mathcal{A}_\nu]_i{}^j \tag{4.58}$$

Decomposition of gluon fields

The hermitian matrix of the gluon field can be decomposed into the basis of the $SU(3)$ generators:

$$\mathcal{A}^j_{i,\mu} = \sum_a A^a_\mu T^{a,j}_i \tag{4.59}$$

In perturbative QCD one usually employs the description in terms of the eight fields A^a_μ, but sometimes the formulation in terms of the matrix $\mathcal{A}$ is useful as well.

Similarly the field-strength tensor is decomposed as

$$\mathcal{F}_{\mu\nu} = T^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + \mathrm{i} g_s [T^a, T^b] A^a_\mu A^b_\nu \equiv T^a F^a_{\mu\nu} \tag{4.60}$$

with

$$F^a_{\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g_s f^{abc} A^b_\mu A^c_\nu \tag{4.61}$$

Infinitesimal transformations

Using the representation of the $SU(3)$ elements in terms of generators, one finds the infinitesimal gauge transformations of the quarks:

$$\begin{aligned}\Delta Q(x) = Q'(x) - Q(x) &= \exp(-\mathrm{i}g_s\omega^a(x)T^a)Q(x) - Q(x)\\ &= -\mathrm{i}g_s\omega^a(x)T^aQ(x) + \mathcal{O}(\omega^2)\end{aligned} \tag{4.62}$$

where the factor of g_s has been introduced as a convention.

From (4.55) we can obtain the transformation of the gluon fields under infinitesimal transformations

$$\begin{aligned} A_\mu^{a\prime}T^a &= UA_\mu^aT^aU^\dagger + \frac{\mathrm{i}}{g_s}(\partial_\mu U)U^\dagger\\ &= (1-\mathrm{i}g_s\omega^bT^b+\dots)A_\mu^aT^a(1+\mathrm{i}g_s\omega^cT^c+\dots) + (\partial_\mu\omega^a)T^a(1+\dots)\\ &= (A_\mu^a + \partial_\mu\omega^a)T^a + A_\mu^a\mathrm{i}g_s\omega^b[T^a,T^b] + \mathcal{O}(\omega^2)\\ &= \left(A_\mu^a + \partial_\mu\omega^a + g_sf^{abc}\omega^bA_\mu^c\right)T^a + \mathcal{O}(\omega^2)\\ \Rightarrow\quad \Delta A_\mu^a &= \partial_\mu\omega^a + g_sf^{abc}\omega^bA_\mu^c\end{aligned} \tag{4.63}$$

Remarks:

- For $g_s \to 0$ the gauge transformation reduces to that of the photon field (3.110).
- for constant transformations $\omega^a \neq \omega^a(x)$ the transformation is analogous to that of the quarks with the generators replaced by that of the adjoint representation:

$$\Delta A_\mu^a|_{\omega=\text{const.}} = -\mathrm{i}g_s\underbrace{(T^{(\mathrm{ad})b})_{ac}}_{-\mathrm{i}f^{bac}=\mathrm{i}f^{abc}}\omega^bA_\mu^c \tag{4.64}$$

- The gauge transformation of the gluon field can be written in terms of the covariant derivative for a field in the adjoint representation:

$$\Delta A_\mu^a = \left(\partial_\mu\delta_{ab} + \mathrm{i}g_s(T^{(\mathrm{ad})c})_{ab}A_\mu^c\right)\omega^b \equiv D_{ab,\mu}^{\mathrm{ad}}\omega^b \tag{4.65}$$

4.2.2 QCD Lagrangian

To form a kinetic Lagrangian for the gluon field we note that the expression

$$\mathrm{tr}[\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}] = \mathcal{F}_{i\,\mu\nu}^{\,j}\mathcal{F}_{j}^{\mu\nu\,i} = \frac{1}{4}F_{\mu\nu}^aF^{a,\mu\nu} \tag{4.66}$$

is gauge invariant because of the cyclicity of the trace and the unitarity of the gauge transformations:

$$\mathrm{tr}[\mathcal{F}'_{\mu\nu}\mathcal{F}^{\mu\nu\prime}] = \mathrm{tr}[U\mathcal{F}_{\mu\nu}U^\dagger U\mathcal{F}^{\mu\nu}U^\dagger] = \mathrm{tr}[\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}U^\dagger U] = \mathrm{tr}[\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}] \tag{4.67}$$

Inserting the explicit expression of the field strength tensor, one sees that the gluon Lagrangian is considerably more complicated because of the commutator terms

$$\begin{aligned}\frac{1}{2}\,\mathrm{tr}[\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}] = \mathrm{tr}\Big[&(\partial_\mu\mathcal{A}_\nu)(\partial^\mu\mathcal{A}^\nu_\mu) - (\partial_\mu\mathcal{A}_\nu)(\partial^\nu\mathcal{A}^\mu_\mu)\\ &+ \mathrm{i}g_s(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu)[\mathcal{A}^\mu,\mathcal{A}^\nu]\\ &+ \frac{(ig_s)^2}{2}[\mathcal{A}_\mu,\mathcal{A}_\nu][\mathcal{A}^\mu,\mathcal{A}^\nu]\Big]\end{aligned} \tag{4.68}$$

The first line is analogous to the free photon Lagrangian in QED, the second line gives a cubic gluon self-interaction term, the last line a quartic self-interaction which lead to Feynman rules of the form (4.4).

The QCD Lagrangian can now be obtained as the generalization of the QED Lagrangian (3.106):

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2}\,\mathrm{tr}[\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}] + \bar{Q}\,(\mathrm{i}\not{D} - m_q)\,Q \tag{4.69}$$

In terms of the gluon fields A^a, the QCD Lagrangian (4.69) can be written as

$$\begin{aligned}\mathcal{L}_{\mathrm{QCD}} &= -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} + \bar{Q}\,(\mathrm{i}(\not{\partial} + \mathrm{i}g_sT^a\not{A}^a) - m_q)\,Q\\ &= \mathcal{L}_{\mathrm{QCD},0} + \mathcal{L}_{\mathrm{QCD},I}\end{aligned} \tag{4.70}$$

with the free Lagrangian

$$\mathcal{L}_{\mathrm{QCD},0} = -\frac{1}{2}\,[(\partial_\mu A^a_\nu)(\partial^\mu A^{a,\nu}) - (\partial_\mu A^a_\nu)(\partial^\nu A^{a,\mu})] + \bar{Q}\,(\mathrm{i}\not{\partial} - m_q)\,Q \tag{4.71}$$

and the interaction

$$\mathcal{L}_{\mathrm{QCD},I} = -g_s\bar{Q}^i\not{A}^aT^{a,j}_iQ_j + g_sf^{abc}(\partial_\mu A_{\nu,a})A^\mu_bA^\nu_c - \frac{g_s^2}{4}f^{abe}f^{cde}A^a_\mu A^b_\nu A^{c,\mu}A^{d,\nu} \tag{4.72}$$

4.2.3 Gauge fixing

In order to derive the gluon propagator, as in QED a gauge-fixing term has to be added to the Lagrangian. Again, usually a covariant term

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi}(\partial_\mu A^{a,\mu})^2 \tag{4.73}$$

is added. It turns out that for calculations beyond tree-level another term has to be added to the Lagrangian if the covariant gauge fixing is used. This so-called **Fadeev-Popov** term is usually derived in the functional integral quantization method that is not discussed in this Lecture. In the Fadeev-Popov method, so-called **ghost fields** c_a and anti-ghost fields $\bar{c}_a$ are introduced: scalar fields in the adjoint representation of $SU(3)$, which, however, are assigned Fermi-statistic. These fields never appear as external states, so the spin-statistics theorem is not violated by the wrong statistics. The Fadeev-Popov Lagrangian is given by

$$\mathcal{L}_{\mathrm{FP}} = (\partial^\mu \bar{c}_a) D^{(\mathrm{ad})}_{ab,\mu} c_b = (\partial^\mu \bar{c}_a)(\partial_\mu \delta_{ab} + g_s f^{abc} A_{c,\mu}) c_b \tag{4.74}$$

The gauge-fixed Lagrangian with the ghost terms is not gauge invariant, but invariant under a global symmetry including the ghost fields, the so-called Becchi-Rouet-Stora-Tyutin (BRST) symmetry. This symmetry can be used to show the independence of scattering amplitudes from the gauge parameter ξ and derive the generalizations of the Ward identities, so-called Slavnov-Taylor identities. A review of BRST symmetry and Slavnov Taylor identities is given in Appendix A

Alternatively, an axial gauge-fixing term could be used,

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi}(n^\mu A^a_\mu)^2. \tag{4.75}$$

In axial gauges, it can be shown that the ghost fields are not necessary.

More generally, one can consider a gauge fixing term of the form

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi}(f^a[\mathcal{A}_\mu])^2 \tag{4.76}$$

with some gauge-fixing functional $f^a[\mathcal{A}_\mu]$. The Fadeev-Popov Lagrangain then involves the gauge variation of the gauge-fixing term:

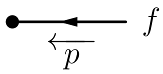
$$\mathcal{L}_{\mathrm{FP}} = -\int \mathrm{d}^4 y\, \bar{c}^a(x) \frac{\delta f^a[\mathcal{A}'_\mu(x)]}{\delta\omega^b(y)} c^b(y) = \bar{c}^a(x)\mathcal{M}^{ab}c^b(x) \tag{4.77}$$

where the last expression holds for a local gauge-fixing functional. For the covariant gauge fixing one obtains the previous result:

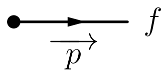
$$f^a = \partial_\mu A^{a,\mu} \qquad \Rightarrow \qquad \mathcal{M}^{ab} = \partial_\mu(\partial^\mu \delta^{ab} + g_s f^{abc} A^\mu_c) \tag{4.78}$$

4.3 Feynman rules

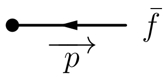
External lines:



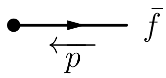
$u_\sigma(p)\,,$



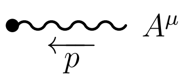
$\bar{u}_\sigma(p)$



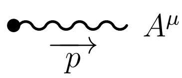
$v_\sigma(p)\,,$



$\bar{v}_\sigma(p)$

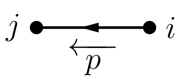


$\epsilon^\mu_\lambda(p)\,,$

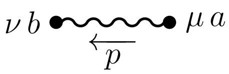


$\epsilon^{\mu *}_\lambda(p)$ (4.79)

Propagators:

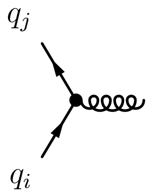


$$\frac{\mathrm{i}\delta_{ij}}{\not{p} - m_f + \mathrm{i}\epsilon}\,,$$

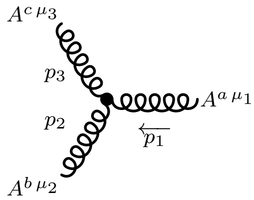


$$\frac{\mathrm{i}\delta_{ab}}{p^2+\mathrm{i}\epsilon}\left[-g^{\mu\nu}+\frac{\mathrm{i}p^\mu p^\nu}{(p^2+\mathrm{i}\epsilon)^2}(1-\xi)\right] \tag{4.80}$$

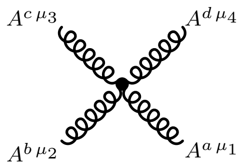
Vertices



$$: -\,\mathrm{i}g_s T_j^{a,i}\gamma^\mu\,, \tag{4.81}$$

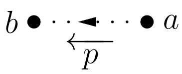


$$: -\,g_s f^{abc}\Big[g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3}+g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1}+g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}\Big] \tag{4.82}$$

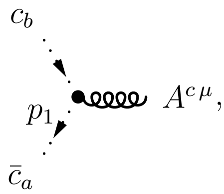


$$\begin{aligned}: -\,\mathrm{i}g_s^2\Big[&f^{abe}f^{cde}(g^{\mu_1\mu_3}g^{\mu_2\mu_4}-g^{\mu_2\mu_3}g^{\mu_1\mu_4})+f^{ace}f^{bde}(g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_2\mu_3}g^{\mu_1\mu_4})\\&+f^{ade}f^{bce}(g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_4\mu_2}g^{\mu_3\mu_1})\Big]\end{aligned} \tag{4.83}$$

Here all momenta are incoming by convention. For one-loop calculation one also needs the Feynman rules for the ghosts resulting from the Fadeev-Popov Lagrangian (4.74)



$$: \frac{\mathrm{i}\delta_{ab}}{p^2+\mathrm{i}\epsilon}\quad ,$$



$$g_s f^{abc}p_1^\mu \tag{4.84}$$

Here p_1 is the incoming anti-ghost momentum. For each closed ghost loop, the diagram has to be multiplied by (-1).

The vertex functions can be obtained from the interaction Lagrangian using the definition (3.159). For instance, the three gluon vertex is obtained from

$$\mathrm{i}\int \mathrm{d}^4x\,\langle 0|\mathcal{L}_I(x)|g_a^{p_1}g_b^{p_2}g_c^{p_3}\rangle = (2\pi)^4\delta^4(\sum_i p_i)V^{(3)\,\mu_1\mu_2\mu_3}_{g_a g_b g_c}\epsilon_{\mu_1}(p_1)\epsilon_{\mu_2}(p_2)\epsilon_{\mu_3}(p_3) \tag{4.85}$$

There are six possibilities to contract the fields with the external states. One example looks like

$$\begin{aligned}&\mathrm{i}g_s f^{lmn}g_{\rho\mu}g_{\nu\sigma}(\partial^\rho A_l^\sigma(x))A_m^\mu(x)A_n^\nu(x)|g_a^{p_1}g_b^{p_2}g_c^{p_3}\rangle\\&\qquad\sim(\mathrm{i}g_s)f^{abc}g_{\rho\mu}g_{\nu\sigma}(-\mathrm{i}p_1^\rho)g^{\mu_1\sigma}g^{\mu_2\mu}g^{\mu_3\nu}\epsilon_{\mu_1}(p_1)\epsilon_{\mu_2}(p_2)\epsilon_{\mu_3}(p_3)\end{aligned} \tag{4.86}$$

where we dropped the exponentials leading to the delta function. Adding the other contributions gives the vertex function

$$\begin{aligned} V_{ggg}^{\mu_1\mu_2\mu_3} &= g_s\Big[f^{abc}p_1^{\mu_2}g^{\mu_3\mu_1} + f^{acb}p_1^{\mu_3}g^{\mu_2\mu_1} \\ &\qquad + f^{bac}p_2^{\mu_1}g^{\mu_2\mu_3} + f^{bca}p_2^{\mu_3}g^{\mu_1\mu_2} + f^{cab}p_3^{\mu_1}g^{\mu_2\mu_3} + f^{cba}p_3^{\mu_2}g^{\mu_3\mu_1}\Big] \\ &= -g_s f^{abc}\Big[g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3} + g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}\Big] \end{aligned} \tag{4.87}$$

where the antisymmetry of the structure constants has been used in the last step.

In the four-gluon vertex there are $4\times3\times2\times1 = 24$ possible contractions. A typical contribution is

$$\begin{aligned} &-\mathrm{i}\frac{g_s^2}{4}f^{lmn}f^{opn}g_{\rho\mu}g_{\nu\sigma}A_l^\mu(x)A_m^\nu(x)A_o^\rho(x)A_p^\sigma(x)|g_a^{p_1}g_b^{p_2}g_c^{p_3}g_d^{p_4}\rangle \\ &\qquad\qquad \Rightarrow -\mathrm{i}\frac{g_s^2}{4}f^{abe}f^{cde}g^{\mu_1\mu_3}g^{\mu_2\mu_4}\epsilon_{\mu_1}(p_1)\epsilon_{\mu_2}(p_2)\epsilon_{\mu_3}(p_3)\epsilon_{\mu_4}(p_4) \end{aligned} \tag{4.88}$$

Adding all contributions gives the four-point vertex

$$\begin{aligned} V_{gggg}^{\mu_1\mu_2\mu_3\mu_4} = -\mathrm{i}g_s^2\Big[&f^{abe}f^{cde}(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_2\mu_3}g^{\mu_1\mu_4}) + f^{ace}f^{bde}(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_2\mu_3}g^{\mu_1\mu_4}) \\ &+ f^{ade}f^{bce}(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_4\mu_2}g^{\mu_3\mu_1})\Big] \end{aligned} \tag{4.89}$$

where the factor of $1/4$ has canceled against the twenty-four possible contractions, leaving six different structures in the vertex.

4.4 Evaluation of colour factors

An additional complication of calculation in QCD compared to QED is the presence of the $SU(3)$ generators T^a and the structure constant f^{abc} in the Feynman rules. Since the colour quantum number is not observed, one sums over final-state colour and averages over initial-state colour, analogously to the treatment of spin for unpolarized cross sections. These sums lead to the appearance of colour factors in QCD calculations, that we will define here. We will initially consider the group $SU(N_c)$ and set the number of colours $N_c \to 3$ in the end.

Casimir operators

The operator T^aT^a commutes with all generators:

$$[(T^aT^a), T^b] = T^a[T^a, T^b] + [T^a, T^b]T^a = \mathrm{i}f^{abc}\left(T^aT^c + T^cT^a\right) = 0 \tag{4.90}$$

since the structure constants are antisymmetric. An operator that commutes with all generators is called a **Casimir operator**. It can be shown that a Casimir operator is proportional to the unit matrix in an irreducible representation,

$$(\mathbf{T}^{(R)a}\mathbf{T}^{(R)a}) = C_R\mathbf{1} \tag{4.91}$$

where usually the number C_R is called the **quadratic Casimir of the representation** R.

Comparing to the definition of the index of a representation (4.92),

$$\mathrm{tr}(\mathbf{T}^{(R)a}\mathbf{T}^{(R)b}) = T_R\delta_{ab}. \tag{4.92}$$

one has the relation

$$\mathrm{tr}(\mathbf{T}^{a(R)a}\mathbf{T}^{(R)a}) = C_R \dim(R) = T_R \dim(G) \tag{4.93}$$

This can be used to compute the quadratic Casimir of the fundamental representation, $C_3 \equiv C_F$:

$$C_F N_c = T_F \dim(G) = \frac{1}{2}(N_c^2 - 1) \tag{4.94}$$

i.e.

$$C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3} \tag{4.95}$$

For the adjoint representation one obtains

$$C_A = T_A \tag{4.96}$$

The explicit value is given by ($\Rightarrow$ exercise)

$$C_A = N_c = 3 \tag{4.97}$$

An alternative derivation follows from a general relation for quadratic Casimir operators that can be obtained from the decomposition of the tensor product

$$R \otimes R' = R_1 \oplus R_2 \cdots = \sum_\alpha R_\alpha \tag{4.98}$$

and the expression (4.47) for the generator in the product representation:

$$\mathbf{T}^{(R\otimes R')} = \mathbf{T}^{(R)} \otimes \mathbf{1}^{(R')} + \mathbf{1}^{(R)} \otimes \mathbf{T}^{(R')} \tag{4.99}$$

Therefore

$$\mathrm{tr}[\mathbf{T}^{(R\otimes R')}\mathbf{T}^{(R\otimes R')}] = (C_R + C_{R'})\underbrace{\mathrm{tr}[\mathbf{1}^{(R')} \otimes \mathbf{1}^{(R')}]}_{\dim(R)\times\dim(R')} + 2\underbrace{\mathrm{tr}[\mathbf{T}^{(R)} \otimes \mathbf{T}^{(R')}]}_{\mathrm{tr}[\mathbf{T}^{(R)}]\,\mathrm{tr}[\mathbf{T}^{(R')}]=0} \tag{4.100}$$

since the generators are traceless. On the other hand the direct sum of representations on the right-hand side of (4.98) has the generator

$$\begin{pmatrix} \mathbf{T}^{(R_1)} & 0 & \dots & & \\ 0 & \mathbf{T}^{(R_2)} & 0 & \dots & \\ 0 & 0 & \ddots & 0 & \dots \end{pmatrix} \tag{4.101}$$

so that

$$\mathrm{tr}[\mathbf{T}^{(R\otimes R')}\mathbf{T}^{(R\otimes R')}] = \sum_\alpha \mathrm{tr}\,\mathbf{T}^{(R_\alpha)^2} = \sum_\alpha C_{R_\alpha}\dim(R_\alpha) = (C_R + C_{R'})\dim(R)\dim(R') \tag{4.102}$$

This allows to compute the quadratic Casimir of a representation in the decomposition into irreducibles, if all the others are known. The simplest example is the decomposition $3 \otimes \bar{3} = 1 \oplus 8$. Since the group acts trivially on the singlet representation, the generators are vanishing and therefore $C_1 = 0$. This gives for the quadratic Casimir of the adjoint, $C_8 \equiv C_A$:

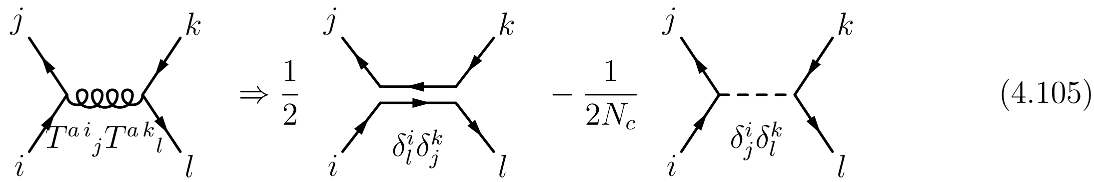
$$C_A = \frac{2C_F}{\dim(8)} \dim(3)\dim(\bar{3}) = 3 = N_c \tag{4.103}$$

Colour Fierz identity

Strings of generators can be simplified using the colour Fierz identity (derived as homework):

$$T^{a,i}{}_j T^{a,k}{}_l = \frac{1}{2}\left(\delta^i{}_l \delta^k{}_j - \frac{1}{N_c}\delta^i{}_j \delta^k{}_l\right) \tag{4.104}$$

Sometimes it is useful to represent this identity graphically:

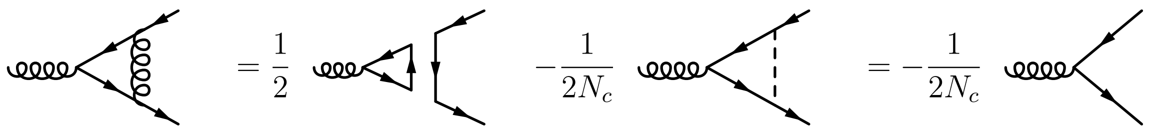


These diagrams are similar to Feynman diagrams but only denote the $SU(3)$ structure, not the propagators, Dirac matrices and spinors.

As an application, we can simplify a string of three generators:

$$(T^a T^b T^a)^i_j = T^{b,k}_l \underbrace{T^{a,i}_k T^{a,l}_j}_{=\frac{1}{2}(\delta^i_j \delta^l_k - \frac{1}{N_c}\delta^i_k \delta^l_j)} = -\frac{1}{2N_c} T^{b,i}_j \tag{4.106}$$

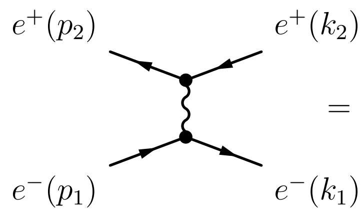
where the tracelessness of the generators was used in the last step. This derivation can be represented diagrammatically as



4.5 Examples

4.5.1 Quark-antiquark potential

Electron-positron scattering in QED

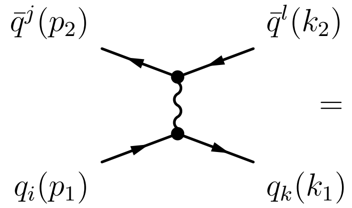


$$= \frac{-\mathrm{i}(\mathrm{i}e)^2}{(p_1 - k_1)^2} (\bar{v}(p_2)\gamma^\mu v(k_2))(\bar{u}(k_1)\gamma^\mu v(p_1)) \tag{4.107}$$

It can be shown that in the non-relativistic limit this expression can be interpreted as the quantum-mechanical amplitude for scattering in an attractive Coulomb potential in momentum space (see e.g. [3]):

$$V(\vec{q}) = \frac{-e^2}{|\vec{q}|^2}. \tag{4.108}$$

As the analogous case in QCD, consider the amplitude for quark-antiquark scattering:



$$= \frac{-\mathrm{i}(\mathrm{i}g_s)^2}{(p_1-k_1)^2}(\bar{v}(p_2)T_j^{a,l}\gamma^\mu v(k_2))(\bar{u}(k_1)T_k^{a,i}\gamma_\mu v(p_1)) \tag{4.109}$$

Comparing to the QED case, we see that in the non-relativistic limit there is a colour-dependent quark-antiquark potential

$$V_{kj}^{li}(\vec{q}) = \underbrace{T_j^{a,l}T_k^{a,i}}_{\equiv C_{kj}^{li}} \frac{-g_s^2}{|\vec{q}|^2} \tag{4.110}$$

There are two possible colour states of $q\bar{q}$ pair since $3 \otimes \bar{3} = 1 \oplus 8$. In the singlet state, the quark-antiquark wave-function is given by

$$|q\bar{q}\rangle_S \propto \delta_j^i \, |q_i \bar{q}^j\rangle\,, \tag{4.111}$$

up to normalization. The potential for the singlet state is therefore obtained by contracting the indices of the incoming quark-antiquark pair:

$$C_{kj}^{li}\delta_i^j = (T^aT^a)_k^l = C_F\delta_k^l \tag{4.112}$$

The colour octet state of the quark-antiquark pair is of the form

$$|q\bar{q}\rangle_8^a \propto T_j^{a,i}\,|q_i\bar{q}^j\rangle \tag{4.113}$$

This form of the state can be understood from the fact that a quark-antiquark pair in an octet state can be produced from a gluon splitting $g^a \to q_i\bar{q}^j$ where the quark gluon vertex involves the generator T_j^{ai}. The potential for the octet state is obtained as

$$C_{kj}^{li}T_j^{a,i} = (T^bT^aT^b)_k^l = -\frac{1}{2N_c}T_k^{a,l} \tag{4.114}$$

Therefore the potential for a quark-antiquark pair in the representation R can be written as

$$V^{(R)}(\vec{q}) = \frac{-g_s^2}{|\vec{q}|^2}C^{(R)} \tag{4.115}$$

with

$$C^{(R)} = \begin{cases} C_F & R = 1\,(\text{attractive}) \\ -\frac{1}{2N_c} & R = 8\,(\text{repulsive}) \end{cases} \tag{4.116}$$

For the singlet representation the potential has the same sign as the electron-positron potential in QED, i.e. it is attractive, while the potential for colour-octet states is repulsive. This is consistent with the observation that only colour-singlet bound states are observed in nature (although of course it is far from a proof).

For bound-states of bottom quarks and for the production of non-relativistic top quarks, the above potential can be used for the (leading-order) computation of the bound-state spectrum or the scattering amplitudes.

Formally one can introduce projectors on the colour singlet and octet states

$$\begin{aligned} P^{(1),li}_{kj} &= \frac{1}{N_c}\delta^l_k\delta^i_j \\ P^{(8),li}_{kj} &= \frac{1}{\underbrace{T_F}_{=2}}\,T^{al}_k T^{ai}_j \end{aligned} \tag{4.117}$$

These are projectors normalized such that $P^{(R)}P^{(R')} = \delta_{RR'}$ and satisfy

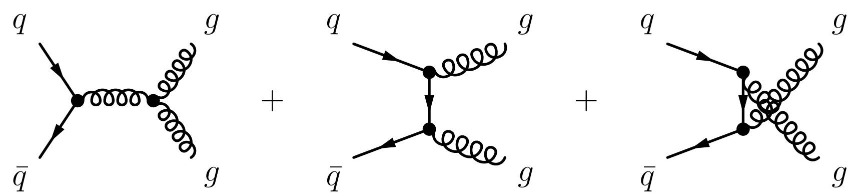
$$P^{(8),li}_{kj} + P^{(1),li}_{kj} = \delta^l_j\delta^i_k \tag{4.118}$$

as can be seen using the colour-Fierz identity. Inserting this resolution of the identity, the potential can be decomposed into the singlet and octet parts:

$$V(\vec{q}) = \sum_R V^{(R)}(\vec{q})P^{(R)} \tag{4.119}$$

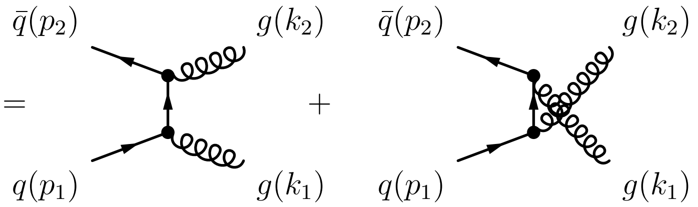
4.5.2 $q\bar{q} \to gg$: Gauge invariance and ghosts

The partonic scattering $q\bar{q} \to gg$ is analogous to the QED process $q\bar{q} \to \gamma\gamma$, but includes an additional diagram because of the three-gluon vertex:



The t- and u-channel diagrams differ from the QED counterparts only by the colour matrices:

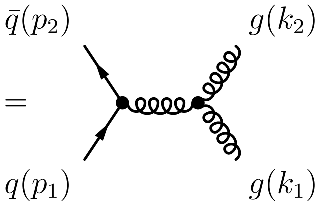
$\mathrm{i}\mathcal{M}_{t+u} =$



$$= (\mathrm{i}g_s)^2\bar{v}(p_2)\left[T^{b,k}_j \not{\epsilon}^*_2 \frac{i}{\not{p}_1 - \not{k}_1 - m_e} T^{a,i}_k \not{\epsilon}^*_1 + \not{\epsilon}^*_1 T^{a,k}_j \frac{i}{\not{p}_1 - \not{k}_2 - m_e} T^{b,i}_k \not{\epsilon}^*_2\right] u(p_1) \tag{4.120}$$

The s-channel diagram reads

$\mathrm{i}\mathcal{M}_s =$



$$
\begin{aligned}
&= \bar{v}(p_2)(-\mathrm{i}g_s)T_j^{c,i}\gamma^\mu u(p_1)\frac{-\mathrm{i}}{(p_1+p_2)^2}\times \\
&\quad g_s f^{abc}[(\epsilon_1^*\cdot\epsilon_2^*)(k_1-k_2)^\mu + \epsilon_2^{\mu *}(2k_2+k_1)\cdot\epsilon_1^* - \epsilon_1^{\mu *}(2k_1+k_2)\cdot\epsilon_2^*]] \\
&= \bar{v}(p_2)(-\mathrm{i}g_s)T_j^{c,i}\gamma^\mu u(p_1)\frac{-\mathrm{i}}{(p_1+p_2)^2}\times \\
&\quad g_s f^{abc}[(\epsilon_1^*\cdot\epsilon_2^*)(k_1-k_2)^\mu + 2\epsilon_2^{\mu *}(k_2\cdot\epsilon_1^*) - 2\epsilon_1^{\mu *}(k_1\cdot\epsilon_2^*)]]
\end{aligned}
\tag{4.121}
$$

Here we have used the transversality of polarization vectors $\epsilon_i \cdot k_i = 0$, and momentum conservation to simplify the three-gluon vertex.

Ward identity

As in QED (3.192), the amplitude must satisfy a Ward identity

$$
\mathcal{M} \;=\; \epsilon_\lambda^\mu(p)^{(*)}\,\tilde{\mathcal{M}}_\mu(p) \quad\Rightarrow\quad p^\mu\,\tilde{\mathcal{M}}_\mu(p) \;=\; 0, \tag{4.122}
$$

Following the discussion given for $e^-e^+ \to \gamma\gamma$ (3.194) one finds for the two quark-exchange diagrams

$$
\begin{aligned}
\mathrm{i}\tilde{\mathcal{M}}_{t+u,\mu}k_1^\mu &= (-\mathrm{i}g_s)^2\bar{v}(p_2)\mathrm{i}\not{\epsilon}_2^*\left[-(T^bT^a)^i_j + (T^aT^b)^i_j\,\right]u(p_1) \\
&= -(-\mathrm{i}g_s)^2 f^{abc}T_j^{c,i}\,\bar{v}(p_2)\not{\epsilon}_2^* u(p_1)
\end{aligned}
\tag{4.123}
$$

The s-channel gluon exchange diagram gives

$$
\begin{aligned}
\mathrm{i}\tilde{\mathcal{M}}_{s,\mu}k_1^\mu &= \bar{v}(p_2)(-\mathrm{i}g_s)T_j^{c,i}\gamma^\mu u(p_1)\frac{-\mathrm{i}}{(p_1+p_2)^2}\times \\
&\quad g_s f^{abc}[-(k_1\cdot\epsilon_2^*)(k_1+k_2)^\mu + 2\epsilon_2^{\mu *}(k_2\cdot k_1)]] \\
&= (-\mathrm{i}g_s)^2 f^{abc}T_j^{c,i}\bar{v}(p_2)\left[-\frac{(\not{p}_1+\not{p}_2)}{2(p_1\cdot p_2)}(k_1\cdot\epsilon_2^*) + \not{\epsilon}_2^*\right]u(p_1) \\
&= (-\mathrm{i}g_s)^2 f^{abc}T_j^{c,i}\bar{v}(p_2)\not{\epsilon}_2^* u(p_1) \\
&= -\mathrm{i}\tilde{\mathcal{M}}_{t+u,\mu}k_1^\mu
\end{aligned}
\tag{4.124}
$$

where momentum conservation and the Dirac equation was used. Therefore the Ward identity is satisfied if all three diagrams are added up:

$$
k_1^\mu\tilde{\mathcal{M}}_{s+t+u,\mu} = 0. \tag{4.125}
$$

- For the Ward identity to hold, it is necessary that the same coupling constant g_s appears in the quark-gluon and the triple-gluon vertex, and that the matrices T^a in the quark-gluon vertex satisfy a Lie-algebra with the constants f^{abc} in the three-gluon vertex as structure constants.

  Based on these observations, one can show that non-abelian gauge theories are the only consistent, renormalizable, theories of self-interacting massless vector bosons.

- In the proof of the Ward identity it was necessary to use the Dirac equation for the external quark lines and the transversality of the gluon polarization vector ϵ_2.

- Since the second polarization vector had to be transverse, it is not possible to drop the terms proportional to the momentum in the polarization sum of the gluons for processes with more than one external gluon:

$$\sum_\lambda \epsilon^\mu_\lambda \epsilon^{\nu*}_\lambda = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{(p\cdot n)} \not\Rightarrow -g^{\nu\nu} \tag{4.126}$$

Relation to ghost diagrams

We consider the "reduced" amplitude obtained by removing both of the gluon polarization vectors:

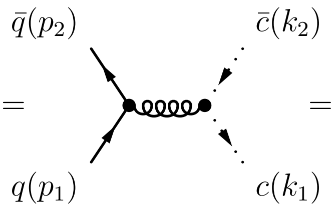
$$\mathcal{M} \;=\; \epsilon^{\mu*}(k_1)\epsilon^{\nu*}(k_2)\,\tilde{\mathcal{M}}_{\mu\nu} \tag{4.127}$$

The only modification in the previous check of the Ward identity appears in the s-channel gluon diagram where the term previously omitted because of $k_2 \cdot \epsilon_2 = 0$ has to be kept:

$$\mathrm{i}\tilde{\mathcal{M}}_{\mu\nu}k_1^\mu = \bar{v}(p_2)(-\mathrm{i}g_s)T_j^{c,i}\gamma^\mu u(p_1)\frac{-\mathrm{i}}{(p_1+p_2)^2}(-g_s f^{abc}k_1^\mu k_{2,\nu}) \tag{4.128}$$

From the ghost Feynman rules one finds the matrix element for the (unphysical) "process" $q\bar{q} \to \bar{c}^a(k_1)c^b(k_2)$:

$\mathrm{i}\mathcal{M}_{c_{k_1}\bar{c}_{k_2}} =$



$$= \bar{v}(p_2)(-\mathrm{i}g_s)T_j^{c,i}\gamma^\mu u(p_1)\frac{-\mathrm{i}}{(p_1+p_2)^2}(-g_s f^{bac}k_1^\mu) \tag{4.129}$$

Therefore one finds that the violation of the WI for the reduced amplitude is proportional to the ghost diagram:

$$\tilde{\mathcal{M}}_{\mu\nu}k_1^\mu = -k_{2,\nu}\mathcal{M}_{c_{k_1}\bar{c}_{k_2}} \tag{4.130}$$

This identity can be derived from the BRST invariance of the gauge-fixed Lagrangian mentioned above. Similarly one finds

$$\tilde{\mathcal{M}}_{\mu\nu}k_2^\nu = -k_{1,\nu}\mathcal{M}_{c_{k_2}\bar{c}_{k_1}} \tag{4.131}$$

Using the relation to ghost diagrams, the sum over gluon polarizations can be simplified according to

$$\begin{aligned}
\sum_{\lambda_1\lambda_2} \tilde{\mathcal{M}}^{\mu\nu} \epsilon^*_{1,\mu} \epsilon^*_{2,\nu} \epsilon_{1,\rho} \epsilon_{2,\sigma} \tilde{\mathcal{M}}^{\rho\sigma,*} &= \tilde{\mathcal{M}}^{\mu\nu} (-g_{\mu\rho}) \left(-g_{\nu\sigma} + \frac{k_{2,\nu} n_\sigma + n_\nu k_{2\sigma}}{(k_2 \cdot n)} \right) \tilde{\mathcal{M}}^{\rho\sigma,*} \\
&= \tilde{\mathcal{M}}^{\mu\nu} \tilde{\mathcal{M}}^*_{\mu\nu} + \mathcal{M}_{c_{k_2}\bar{c}_{k_1}} \frac{k_{1,\rho} n_\sigma}{(k_2 \cdot n)} \tilde{\mathcal{M}}^{\rho\sigma,*} + \tilde{\mathcal{M}}^{\mu\nu} \frac{n_\rho k_{1,\sigma}}{(k_2 \cdot n)} \mathcal{M}^*_{c_{k_2}\bar{c}_{k_1}} \\
&= \tilde{\mathcal{M}}^{\mu\nu} \tilde{\mathcal{M}}^*_{\mu\nu} - \mathcal{M}_{c_{k_2}\bar{c}_{k_1}} \mathcal{M}^*_{c_{k_1}\bar{c}_{k_2}} - \mathcal{M}_{c_{k_1}\bar{c}_{k_2}} \mathcal{M}^*_{c_{k_2}\bar{c}_{k_1}}
\end{aligned} \tag{4.132}$$

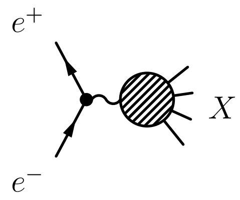
Therefore, instead of replacing $\sum_\lambda \epsilon^\mu_\lambda \epsilon^{\nu *}_\lambda \Rightarrow -g^{\nu\nu}$ as in QED, one either needs to keep the full polarization sum (3.101) or add additional ghost diagrams. The third alternative, advertised already in Section 3.5.2 and developed in Part II, is to compute amplitudes for a given gluon helicity and sum the results over the different helicity combinations.

Chapter 5

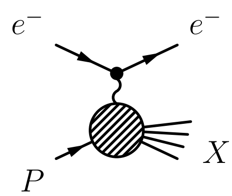
Applications of the Parton model

In this chapter we will first discuss three classic applications of the parton model:

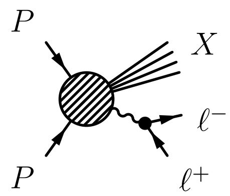
- $e^-e^+ \to$ hadrons



- DIS: $e^-P \to e^- + X$,



- Drell-Yan: $PP \to \ell^-\ell^+ + X$.



These processes are closely related: they are induced by the electromagnetic interaction at leading order, the diagrams are related by crossing, a sum is performed over the unidentified hadronic final states. The processes differ by the presence of one or two identified hadrons in the initial state for DIS and DY.

As an example for partonic scattering processes induced by QCD at leading order, we subsequently discuss dijet production, in particular quark-antiquark scattering.

5.1 $e^-e^+ \to$ Hadrons

As discussed in Section 2.3 the process $e^-e^+ \to$ Hadrons is an important test of QCD and allows to infer the number of quark colours by measuring the so-called R-ratio $\sigma(e^+e^- \to \text{Hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. We here give a field-theoretic derivation of an expression for the total cross section that forms the basis for an analysis in QCD and justifies the naive treatment given in Section 2.3.

5.1.1 Electromagnetic quark current

The interaction Lagrangian of quarks with the electromagnetic field can be written as:

$$\mathcal{L}_{\text{int}} = -e\, A_\mu j_q^\mu \tag{5.1}$$

with the electromagnetic current of the quark fields Q_i with charges q_i:

$$j_q^\mu = \sum_i q_i \bar{Q}_i \gamma^\mu Q_i = \left[-\frac{1}{3}\bar{D}\gamma^\mu D + \frac{2}{3}\bar{U}\gamma^\mu U + \dots \right] \tag{5.2}$$

For expectation values involving quark states, the expectation value of the current is evaluated simply, e.g. at leading order

$$\langle u(k_1)\bar{u}(k_2)|j_q^\mu(x)|0\rangle = \frac{2}{3}(\bar{u}(k_1)\gamma^\mu v(k_2))\, e^{\mathrm{i}(k_1+k_2)\cdot x} \tag{5.3}$$

Expectation values with hadronic states X, $\langle X(p_X)|j_q^\mu(x)|0\rangle$ cannot be evaluated in perturbation theory.

Acting on the current with the translation operator,

$$e^{-\mathrm{i}P\cdot x} j_q^\mu(0) e^{\mathrm{i}P\cdot x} = j_q^\mu(x) \tag{5.4}$$

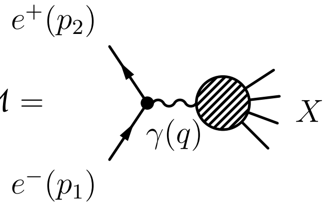
one has the identity

$$\langle X(p_X)|j_q^\mu(x)|Y(p_Y)\rangle = e^{-\mathrm{i}(p_x - p_Y)\cdot x}\, \langle X(p_X)|j_q^\mu(0)|Y(p_Y)\rangle \tag{5.5}$$

5.1.2 Total cross section

At leading order in QED, but all orders in the strong interactions, the matrix element to produce a particular hadronic final state X in e^-e^+ collisions can be written as

$$\mathrm{i}\mathcal{M} =$$



$$= (-\mathrm{i}e)^2\, \langle X|j_{q,\mu}(0)|0\rangle\, \frac{-\mathrm{i}g^{\mu\nu}}{q^2 + \mathrm{i}\epsilon} (\bar{v}_{\lambda_2}(p_2)\gamma_\nu u_{\lambda_1}(p_1)). \tag{5.6}$$

In the full standard model of particle physics, there is also a Z-boson exchange diagram, which will be neglected here. After the same steps as for $e^-e^+ \to \mu^-\mu^+$ one finds for the spin-averaged squared matrix element

$$\overline{|\mathcal{M}_{e^-e^+\to X}|^2} = \frac{e^4}{4s^2} \langle 0|j_{q,\nu}(0)|X(k_X)\rangle \langle X(k_X)|j_{q,\mu}(0)|0\rangle \underbrace{\mathrm{tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu]}_{\equiv 4L^{\mu\nu}} \tag{5.7}$$

with the so-called "leptonic tensor"

$$L^{\mu\nu} = p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - (p_1 \cdot p_2)g^{\mu\nu} \tag{5.8}$$

The lepton tensor is transverse,

$$q_\mu L^{\mu\nu} = p_2^\nu(q \cdot p_1) + p_1^\nu(q \cdot p_2) - (p_1 \cdot p_2)q^\nu = (p_1 \cdot p_2)(p_1^\nu + p_2^\nu - q^\nu) = 0 \tag{5.9}$$

where $q = p_1 + p_2$, and using $p_1^2 = p_2^0$ for massless electrons.

The cross-section summed over all hadronic final states gives

$$\begin{aligned} \sigma &= \frac{1}{2s} \sum_X \int \mathrm{d}\tilde{\phi}_X (2\pi)^4 \delta^4(p_1 + p_2 - k_X) \overline{|\mathcal{M}_{e^-e^+\to X}|^2} \\ &= \frac{e^4}{2s^3} W_{\mu\nu} L^{\mu\nu} \end{aligned} \tag{5.10}$$

with the so-called **hadronic tensor**

$$W_{\mu\nu} = \sum_X \int \mathrm{d}\tilde{\phi}_X (2\pi)^4 \delta^4(q - k_X) \langle 0|j_{q,\nu}(0)|X(k_X)\rangle \langle X(k_X)|j_{q,\mu}(0)|0\rangle \,. \tag{5.11}$$

We have separated the delta function from the phase-space integral by defining $\mathrm{d}\phi_X = \mathrm{d}\tilde{\phi}_X (2\pi)^4 \delta^4(p_1 + p_2 - k_X)$.

Writing

$$(2\pi)^4 \delta^4(q - k_X) = \int \mathrm{d}^4x e^{-\mathrm{i}x\cdot(q-k_X)} \tag{5.12}$$

and using

$$e^{\mathrm{i}x\cdot k_x} \langle 0|j_{q,\nu}(0)|X(k_X)\rangle = \langle 0|j_{q,\nu}(x)|X(k_X)\rangle \tag{5.13}$$

the hadronic tensor can be written as

$$\begin{aligned} W_{\mu\nu}(q) &= \int \mathrm{d}^4x e^{-\mathrm{i}q\cdot x} \sum_X \int \mathrm{d}\phi_X \langle 0|j_{q,\nu}(x)|X(k_X)\rangle \langle X(k_X)|j_{q,\mu}(0)|0\rangle \\ &= \int \mathrm{d}^4x e^{-\mathrm{i}q\cdot x} \langle 0|j_{q,\nu}(x) j_{q,\mu}(0)|0\rangle \,, \end{aligned} \tag{5.14}$$

where the completeness relation of the hadronic states

$$\sum_X \int \mathrm{d}\phi_X |X\rangle \langle X| = 1 \tag{5.15}$$

was used. The expression (5.14) is the basis for a computation of the cross-section for $e^-e^+ \to$ hadrons in QCD. One can argue that for $q \to \infty$ the expectation value of the currents is dominated by the region $x \to 0$ where perturbation theory in QCD is reliable. Note that the dependence on the hadronic states $|X\rangle$ only drops out in the inclusive cross section $e^-e^+ \to$ hadrons where one sums over all hadronic final states.

Current conservation $\partial_\mu j^\mu = 0$ implies that the hadronic tensor satisfies

$$q_\mu W^{\mu\nu}(q) = 0. \tag{5.16}$$

Since it can only depend on q, it must have the structure

$$W^{\mu\nu}(q) = W(q^2)(q^\mu q^\nu - g^{\mu\nu} q^2). \tag{5.17}$$

The cross section becomes

$$\begin{aligned} \sigma &= \frac{e^4}{2s^3} W(q^2) q^2 (-g^{\mu\nu} L_{\mu\nu}) \\ &= \frac{8\pi^2\alpha^2}{s} W(s) \end{aligned} \tag{5.18}$$

since

$$L^\mu_\mu = (2 - g^\mu_\mu)(p_1 \cdot p_2) = -q^2 = -s \tag{5.19}$$

Recall that the the R-ratio has been introduced in (10.112)

$$R = \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma^0(e^+e^- \to \mu^+\mu^-)}. \tag{5.20}$$

Here we normalize by the leading-order cross-section for $e^+e^- \to \mu^+\mu^-$ for $m_\nu \to 0$ (3.171):[1]

$$\sigma^0(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}. \tag{5.21}$$

One therefore gets an expression of the R-ratio to all orders in the strong interactions:

$$R(q^2) = 6\pi W(q^2) = -\frac{2\pi}{q^2} \int \mathrm{d}^4x e^{-\mathrm{i}q\cdot x} \langle 0| j^\mu_q(x) j_{q,\mu}(0)|0\rangle \tag{5.22}$$

For a perturbative evaluation, one can introduce a complete set of **partonic** final states

$$\{|x\rangle\} = \{|q\bar{q}\rangle, |q\bar{q}g\rangle, |q\bar{q}gg\rangle, |q\bar{q}q\bar{q}\rangle, \ldots\} \tag{5.23}$$

$$R(q^2) = 6\pi W(q^2) = -\frac{2\pi}{q^2} \int \mathrm{d}^4x e^{-\mathrm{i}q\cdot x} \sum_x \int \mathrm{d}\Phi_x \langle 0| j^\mu_q(x)|x\rangle \langle x| j_{q,\mu}(0)|0\rangle \tag{5.24}$$

At leading order only the first term $|x\rangle = |q\bar{q}\rangle$ contributes and one recovers the "naive quark model" prediction discussed in Section 2.3.

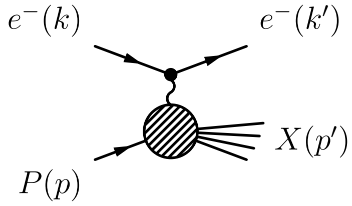
[1] Sometimes the normalization is performed with respect to the full $e^+e^- \to \mu^+\mu^-$ cross section. The use of the LO cross section here corresponds to the PDG convention.

5.2 DIS

We are interested in the "inclusive" cross section for the process

$$e^- p \to e^- X$$

where X denotes the complete hadronic final state.



As sketched in Section 2.2, in the parton model the parton i carries a momentum fraction ξ_i of the proton momentum p:

$$p_i^\mu = \xi_i p^\mu. \tag{5.25}$$

In the naive parton model, the cross section for deep-inelastic electron-proton scattering in the Bjorken limit

$$Q^2 \to \infty \qquad \text{with} \qquad x = \frac{Q^2}{p \cdot q} = \text{fixed} \tag{5.26}$$

is written as a incoherent sum over "partonic cross sections" convoluted with parton distribution functions $f_i(\xi)$

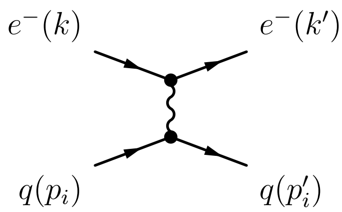
$$\sigma(e^- p \to e^- X) = \int_0^1 d\xi \sum_i f_i(\xi)\, \hat{\sigma}(e^- q_i \to e^- q_i) \tag{5.27}$$

where the sum is over the quark flavours i. We here give a more detailed discussion of DIS in the naive parton model and then perform a general field-theoretical analysis which forms the basis of the treatmet of DIS in QCD. We will parametrize the cross section in terms of **momentum transfer** Q^2 and the dimensionless variables x and y:

$$Q^2 = -(k-k')^2 \qquad\qquad x = \frac{Q^2}{2(p \cdot q)} \qquad\qquad y = \frac{p \cdot (k-k')}{(k \cdot p)} \tag{5.28}$$

DIS cross section in the naive parton model

We first discuss the partonic cross section for quark-electron scattering in order to obtain the parton-model prediction for the cross section (5.27).



The calculation for $e^- q \to e^- q$ is analogous to $e\mu^- \to e^\cdot\mu^-$. The differential cross section can be written as

$$\frac{\mathrm{d}\hat{\sigma}_{eq}}{\mathrm{d}Q^2} = \frac{2\pi\alpha^2 q_i^2}{Q^4}\left[1+(1-y)^2\right] \tag{5.29}$$

The Bjorken variable x is related to the momentum fraction of the quark by

$$x = \frac{Q^2}{2(p\cdot q)} = \xi_i \frac{(k\cdot k')}{(p_i\cdot q)} = \xi_i \tag{5.30}$$

so we can write

$$\frac{\mathrm{d}\hat{\sigma}_{eq}}{\mathrm{d}Q^2\mathrm{d}x} = \frac{2\pi\alpha^2 q_i^2}{Q^4}\left[1+(1-y)^2\right]\delta\left(x-\xi_i\right). \tag{5.31}$$

The parton-model prediction for the double-differential DIS cross section is therefore

$$\frac{\mathrm{d}\sigma_{eP}}{\mathrm{d}Q^2\mathrm{d}x} = \sum_i f_i(x)\,\frac{2\pi\alpha^2 q_i^2}{Q^4}\left[1+(1-y)^2\right], \tag{5.32}$$

where the relation of y to Q^2 is given by

$$y = \frac{(p\cdot q)}{(k\cdot p)} = \frac{(p_i\cdot q)}{(k\cdot p_i)} = \frac{2(p_i\cdot q)}{\hat{s}} = \frac{Q^2}{xs}. \tag{5.33}$$

Details of the calculation:

- Spin averaged matrix element in the limit $m_e, m_q \to 0$

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{e^4 q_i^2}{4Q^4}\,\mathrm{tr}[((\not{p}_i' + m_q)\gamma_\mu(\not{p}_i + m_q)\gamma_\nu]\,\mathrm{tr}[(\not{k} + m_e)\gamma^\mu(\not{k}' + m_e)\gamma^\nu] \\
&= \frac{8e^4 q_i^2}{Q^4}\left[(k'\cdot p_i')(k\cdot p_i) + (k\cdot p_i')(p_i\cdot k')\right] \\
&= \frac{8e^4 q_i^2}{Q^4}(k\cdot p)^2\left[1 + \underbrace{\frac{(k\cdot p_i')(p_i\cdot k')}{(k\cdot p)^2}}_{(1-y)^2}\right]
\end{aligned} \tag{5.34}$$

  where momentum conservation has been used in the form $(k-p_i')^2 = (p_i-k')^2$.

- "Partonic" cross-section:

$$\begin{aligned}
\frac{\mathrm{d}\hat{\sigma}_{eq}}{\mathrm{d}\phi\,\mathrm{d}\cos\theta} &= \frac{1}{2\hat{s}}\frac{1}{8(2\pi)^2}\overline{|\mathcal{M}|^2} \\
&= \frac{\alpha^2 q_i^2}{Q^4}\frac{\hat{s}}{2}\left[1+(1-y)^2\right]
\end{aligned} \tag{5.35}$$

  where the "partonic centre of mass energy"

$$\hat{s} = (p_i + k)^2 = 2(p_i\cdot k) = xs \tag{5.36}$$

  was introduced. In the centre-of-mass frame

$$Q^2 = -(k-k')^2 = 2k\cdot k' = 2E^2(1-\cos\theta) = \frac{\hat{s}}{2}(1-\cos\theta) \tag{5.37}$$

Field-theoretic treatment

We now derive a general parametrization of the DIS cross section, which holds to all orders in the strong interactions and which allows to compute radiative corrections to the parton-model result. In analogy to $e^-e^+ \to$ hadrons, the matrix element for the production of a particular final state X can be written as

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}e)^2 \langle X|j_{q,\mu}(0)|P(p)\rangle \frac{-\mathrm{i}g^{\mu\nu}}{p^2+\mathrm{i}\epsilon}(\bar{u}_{\sigma_1}(k')\gamma_\nu u_{\lambda_1}(k)). \tag{5.38}$$

Squared and spin-averaged matrix element

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4Q^4}\langle P(p)|j_{q,\nu}(0)|X(p_X)\rangle \langle X(p_X)|j_{q,\mu}(0)|P(p)\rangle \underbrace{\mathrm{tr}[\not{k}\gamma^\mu\not{k}'\gamma^\nu]}_{4L^{\mu\nu}} \tag{5.39}$$

where the Lepton tensor is now given by

$$L^{\mu\nu} = k^\mu k'^\nu + k'^\mu k^\nu - (k\cdot k')g^{\mu\nu}. \tag{5.40}$$

The DIS cross section is obtained by summing over all hadronic final states

$$\begin{aligned}\mathrm{d}\sigma &= \frac{1}{2s}\sum_X \mathrm{d}\tilde{\phi}_X\left(\frac{\mathrm{d}^3k'}{(2\pi)^3(2k'^0)}\right)(2\pi)^4\delta(p+k-k'-k_X)\overline{|\mathcal{M}|^2}\\ &= \frac{1}{2s}\left(\frac{\mathrm{d}^3k'}{(2\pi)^3(2k'^0)}\right)\frac{(2\pi)e^4}{Q^4}L^{\mu\nu}W_{\mu\nu}.\end{aligned} \tag{5.41}$$

Here the hadronic tensor has been introduced:

$$\begin{aligned}W_{\mu\nu}(q,p) &= \frac{1}{2\pi}\sum_X \mathrm{d}\tilde{\phi}_X \langle P(p)|j_{q,\nu}(0)|X(k_X)\rangle\langle X(k_X)|j_{q,\mu}(0)|P(p)\rangle (2\pi)^4\delta^4(p+q-k_X)\\ &= \frac{1}{2\pi}\int \mathrm{d}^4x e^{-\mathrm{i}q\cdot x}\sum_X\int \mathrm{d}\phi_X \langle P(p)|j_{q,\nu}(x)|X(k_X)\rangle\langle X(k_X)|j_{q,\mu}(0)|P(p)\rangle\\ &= \frac{1}{2\pi}\int \mathrm{d}^4x e^{-\mathrm{i}q\cdot x}\langle P(p)|j_{q,\nu}(x)j_{q,\mu}(0)|P(p)\rangle\,.\end{aligned} \tag{5.42}$$

As in the case of $e^-e^+ \to$ hadrons, current conservation implies that the hadronic tensor must satisfy $q_\mu W^{\mu\nu}(q,p) = 0$. Further, for the unpolarized case discussed here, the hadronic tensor is symmetric, $W^{\mu\nu}(q,p) = W^{\nu\mu}(q,p)$, as the leptonic tensor. It can be shown ($\Rightarrow$ homework) that the hadronic tensor is determined by two coefficient functions $F_{1/2}$:

$$W_{\mu\nu}(q,p) = F_1\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)+\frac{F_2}{(p\cdot q)}\left(p^\mu - q^\mu\frac{(p\cdot q)}{q^2}\right)\left(p^\nu - q^\nu\frac{(p\cdot q)}{q^2}\right). \tag{5.43}$$

The so-called **structure functions** F_i can depend on the Lorentz invariants $q^2 = -Q^2$, $p^2 = m_P^2$ and $q\cdot p = x\,Q^2$. Suppressing the proton mass, they therefore depend on two parameters, $F_{1/2}(x, Q^2)$. The double-differential cross section expressed in terms of the structure function is found to be

$$\frac{\mathrm{d}\sigma_{eP}}{\mathrm{d}Q^2\mathrm{d}x} = \frac{2\pi\alpha}{Q^4}\frac{1}{x}\left[xy^2F_1(x,Q^2) + F_2(x,Q^2)(1-y)\right]. \tag{5.44}$$

Comparing to the parton model result (5.32) relates the structure functions to the parton distribution functions:

$$F_1(x,Q^2) = x\sum_i q_i^2\, f_i(x)\,, \qquad F_2(x,Q^2) = \frac{1}{2}\sum_i q_i^2\, f_i(x)\,, \tag{5.45}$$

The relation

$$F_2(x,Q^2) = 2xF_1(x,Q^2) \tag{5.46}$$

is called the **Callen-Gross relation**. It results from the spin 1/2 nature of the quarks, since the parton-model result was obtained (5.32) under this assumption. It can be shown that for a scalar parton $F_1 = 0$. Instead of comparing to the cross section in the naive parton model (5.32), the Callen-Gross relation (5.32) and the independence of the structure functions of Q^2 at leading order (Bjorken scaling) can be obtained using an analysis of the hadronic tensor (5.42) and evaluating the operator product $j_q^\mu(x)j_q^\nu(0)$ in the limit $x \to 0$ for free quarks. This framework also allows the computation of radiative corrections in full QCD. Taking these corrections into account, Bjorken scaling and the Callen-Gross relation receive calculable higher-order corrections. In particular, the PDFs have to be taken as scale dependent, $f_i(x, Q^2)$.

Derivation of the cross section

The double-differential cross section with respect to $E' = k'^0$ and $\cos\theta$ can be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E'\mathrm{d}\cos\theta} = \frac{1}{2s}\frac{e^4}{Q^4}\frac{E'}{4\pi}L^{\mu\nu}W_{\mu\nu}. \tag{5.47}$$

In the proton rest frame,

$$Q^2 = 2(k\cdot k') = 2EE'(1-\cos\theta)\,, \qquad x = \frac{Q^2}{2(p\cdot q)} = \frac{EE'(1-\cos\theta)}{m_P(E-E')}. \tag{5.48}$$

Using these expressions, the Jacobian is calculated as

$$\left|\frac{\partial(Q^2,x)}{\partial(E',\cos\theta)}\right| = \frac{2xEE'}{E-E'} = \frac{2x}{y}E', \tag{5.49}$$

since in the rest frame

$$y = \frac{(p\cdot q)}{(k\cdot p)} = \frac{E-E'}{E}. \tag{5.50}$$

The double-differential cross section with respect to the Lorentz invariants x and Q^2 is found to be:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\mathrm{d}x} = \frac{1}{2s}\frac{e^4}{Q^4}\frac{1}{8\pi}\frac{y}{x}L^{\mu\nu}W_{\mu\nu}. \tag{5.51}$$

Evaluating the cross section now in the infinite momentum frame, $m_e^2, m_P^2 \to 0$, the contraction of the leptonic and hadronic tensors is calculated as

$$\begin{aligned} L_{\mu\nu}W^{\mu\nu} &= -q^2 F_1 + \frac{(p\cdot k)(p\cdot k')}{(p\cdot q)} F_2 \\ &= \frac{2s}{y}\left[xy^2 F_1 + F_2(1-y)\right]. \end{aligned} \tag{5.52}$$

We finally obtain the expression for the double-differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\mathrm{d}x} = \frac{2\pi\alpha}{Q^4}\frac{1}{x}\left[xy^2 F_1 + F_2(1-y)\right]. \tag{5.53}$$

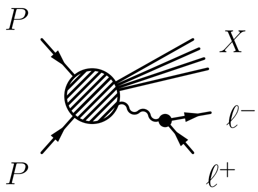
5.3 The Drell-Yan process

A classic example for a scattering process in proton-proton or proton-antiproton collisions is the process of di-lepton production,

$$P(p_1)P(p_2) \to \ell^-(k_1)\ell^+(k_2) + X(k_X) \tag{5.54}$$

Similarly to $e^-e^+ \to$ hadrons, the matrix element can be written in terms of an expectation value of the electromagnetic current:

$\mathrm{i}\mathcal{M} =$



$$= (-\mathrm{i}e)^2(\bar{u}_{\lambda_1}(k_1)\gamma_\nu v_{\lambda_1}(k_2))\frac{-\mathrm{i}g^{\mu\nu}}{q^2+\mathrm{i}\epsilon}\langle X(k_X)|j_{q,\mu}(0)|P(p_1)P(p_2)\rangle \tag{5.55}$$

Following the derivation of the cross sections in the previous examples one can write the differential cross-section in the form

$$\mathrm{d}\sigma = \frac{1}{2s}\prod_{i=1,2}\left(\frac{\mathrm{d}^3k_i}{(2\pi)^3(2k_i{}^0)}\right)\frac{4e^4}{q^4}L^{\mu\nu}W_{\mu\nu} \tag{5.56}$$

where the hadronic and leptonic tensors are now given by

$$\begin{aligned} W_{\mu\nu} &= \int \mathrm{d}^4x e^{-\mathrm{i}(k_1+k_2)\cdot x}\langle P(p_1)P(p_2)|j_{q,\nu}(x)j_{q,\mu}(0)|P(p_1)P(p_2)\rangle, \\ L^{\mu\nu} &= k_1^\mu k_2^\nu + k_2{}^\mu k_1^\nu - (k_1\cdot k_2)g^{\mu\nu}. \end{aligned} \tag{5.57}$$

In the centre-of-mass frame of the lepton pair, one can compute the integral of the leptonic tensor over the relative angle between the leptons:

$$\int \mathrm{d}\phi_{12}\mathrm{d}\cos\theta_{12}L^{\mu\nu} = \frac{4\pi}{3}(-q^2g^{\mu\nu} + q^\mu q^\nu). \tag{5.58}$$

Because of current conservation, only the term proportional to $g^{\mu\nu}$ contributes to the contraction of the hadronic tensor.

This gives the cross section with respect to the invariant mass of the lepton pair $Q^2 = (k_1 + k_2)^2$ as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{1}{s}\frac{\alpha^2}{Q^2}\frac{4\pi}{3}W(Q^2,\tau) \tag{5.59}$$

with

$$\tau = \frac{Q^2}{s} \tag{5.60}$$

and

$$W(Q^2,\tau) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4q\,\delta(q^2 - Q^2)\theta(q^0)(-g^{\mu\nu}W_{\mu\nu}) \tag{5.61}$$

Starting from this expression it has been shown, that in the limit $s \to \infty$, $\tau =$ const. the hadronic tensor factorizes in a convolution of the PDFs $f_i(x)$ and **hard-scattering coefficient functions** H_{ij}:

$$W(Q^2,\tau) = \int \sum_{ij} \frac{\mathrm{d}x_1}{x_1} f_i(x_1) \int \frac{\mathrm{d}x_2}{x_2} f_j(x_2) H_{ij}(\tau/x_1x_2, Q^2) \tag{5.62}$$

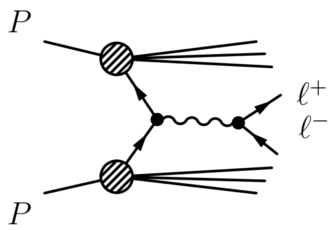
The PDFs are universal, i.e. the same functions as those appearing in DIS. The formula for the cross section can be written in the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \int \mathrm{d}x_1\mathrm{d}x_2 f_i(x_1) f_j(x_2) \frac{\mathrm{d}\hat{\sigma}_{ij}(\ell^+\ell^- + X)}{\mathrm{d}Q^2} \tag{5.63}$$

with the **partonic cross section**

$$\frac{\mathrm{d}\hat{\sigma}_{ij}(\ell^+\ell^- + X)}{\mathrm{d}Q^2} = \frac{1}{\hat{s}}\frac{\alpha^2}{Q^2}\frac{4\pi}{3}H_{ij}(\tau/x_1x_2, Q^2) \tag{5.64}$$

with the **partonic centre-of-mass** energy $\hat{s} = x_1x_2s$. This **factorization** of the cross section can be represented graphically as



At leading order, only initial-state quarks contribute, and lepton pair is produced at a fixed invariant mass $Q^2 = \hat{s}$. The total partonic cross section is identical to that of $e^-e^+ \to \mu^-\mu^+$, up to the quark charge q_q and a factor of $1/N_c$ because of the average of the initial-state quark colours:

$$\frac{\mathrm{d}\hat{\sigma}_{q\bar{q}}(\ell^+\ell^-)}{\mathrm{d}Q^2} = \frac{4\pi\alpha^2 q_q^2}{3Q^2}\frac{1}{N_c}\delta(\hat{s} - Q^2) = \frac{4\pi\alpha^2 q_q^2}{3Q^2}\frac{1}{N_c}\frac{1}{s}\delta(x_1x_2 - \tau) \tag{5.65}$$

From comparison to the general expression for the partonic cross section (5.64) one gets the leading-order expression of the hard coefficient,

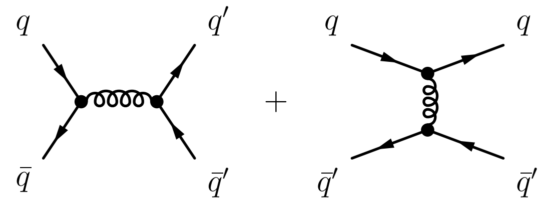
$$H_{q\bar{q}} = \frac{1}{N_c}\tau\delta(x_1x_2 - \tau) \tag{5.66}$$

5.4 Dijet cross sections

There are four classes of partonic processes contributing to "dijet" production at hadron colliders,

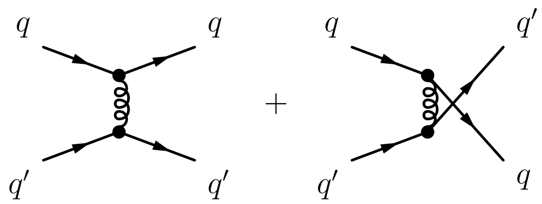
$$pp \to jj, \tag{5.67}$$

$q\bar{q} \to q\bar{q}$



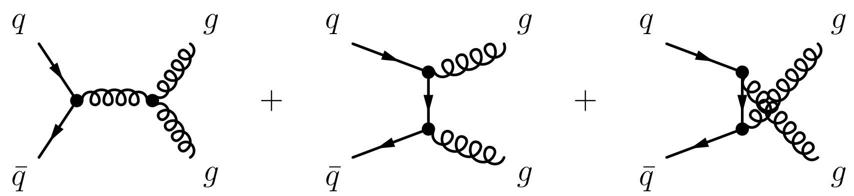
(5.68)

$qq \to qq$



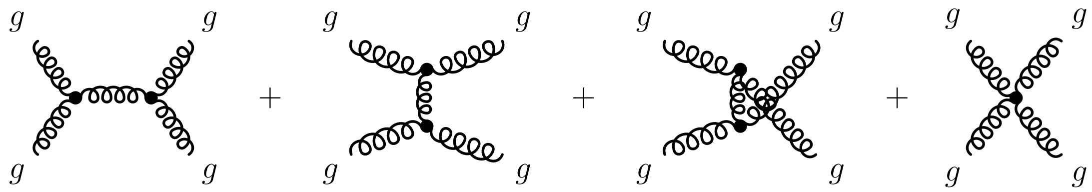
(5.69)

$q\bar{q} \to gg$ and crossed processes $qg \to qg$, $\bar{q}g \to \bar{q}g$, $gg \to \bar{q}g$



(5.70)

$gg \to gg$



(5.71)

The leading-order prediction for dijet production is then obtained by convoluting the partonic cross section with the PDFs and summing over all initial- and final-state partons

$$\mathrm{d}\sigma(pp \to jj) = \int \mathrm{d}x_1 \mathrm{d}x_2 \sum_{i,j,k,l} f_i(x_1) f_j(x_2) \mathrm{d}\hat{\sigma}(ij \to kl) \tag{5.72}$$

where $i, j, k, l \in \{q_i, \bar{q}_i, g\}$.

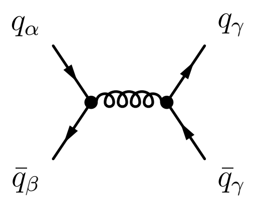
In practice one also needs a precise definition of a "jet" since two partons can become soft or collinear to each other so that they have to be combined into the same jet.

The computation of these cross sections for all channels using the Feynman rules and the textbook methods (squaring the amplitude, computing all interference terms using completeness relations) is "straightforward but tedious". We will only discuss the quark-antiquark subprocesses here in order to illustrate new features of QCD compared to QED. The computation of all cross sections will be much simpler using the spinor-helicity method discussed in Part II

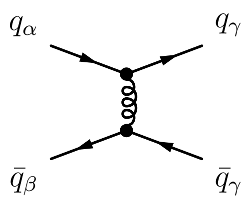
5.4.1 Four-quark processes

The first of the two diagrams in (5.68) can only occur if the flavour of the incoming (outgoing) quark and antiquark are the same, the second one only if the incoming and outgoing quark (antiquark) have the same flavour. Therfore the matrix element can be written in the form

$$\mathcal{M}(q_\alpha \bar{q}_\beta \to q_\gamma \bar{q}_\delta) =$$



$$+\,(-1)$$



(5.73)

$$= \delta_{\alpha\beta}\delta_{\gamma\delta}\mathcal{M}_s - \delta_{\alpha\gamma}\delta_{\beta\delta}\mathcal{M}_t$$

The relative minus sign in the t-channel diagram arises from the exchange of an external fermion line.

The only difference of the s and t-channel matrix element compared to the QED matrix elements for $e^-e^+ \to \mu^-\mu^+$ and $e^-\mu^+ \to e^-\mu^+$ is given by the colour matrices:

$$\begin{aligned}
\mathrm{i}\mathcal{M}_s &= (-\mathrm{i}g_s)^2 (\bar{u}^{j_1}_{\sigma_1}(k_1) T^{a,j_2}_{j_1} \gamma_\mu v_{j_2\sigma_2}(k_2)) \frac{-\mathrm{i}}{(p_1+p_2)^2 + \mathrm{i}\epsilon} (\bar{v}^{i_2}_{\lambda_2}(p_2) T^{a,i_1}_{i_2} \gamma^\mu u_{i_1\lambda_1}(p_1)) \\
&\equiv T^{a,j_2}_{j_1} T^{a,i_1}_{i_2}\, \mathrm{i}\widetilde{\mathcal{M}}_s && (5.74) \\
\mathrm{i}\mathcal{M}_t &= (-\mathrm{i}g_s)^2 (\bar{v}^{i_2}_{\lambda_2}(p_2) T^{a,j_2}_{i_2} \gamma_\mu v_{j_2\sigma_2}(k_2)) \frac{-\mathrm{i}}{(p_1-k_1)^2 + \mathrm{i}\epsilon} (\bar{u}^{j_1}_{\sigma_1}(k_1) T^{a,i_1}_{j_1} \gamma^\mu u_{i_1\lambda_1}(p_1)) \\
&\equiv T^{a,j_2}_{i_2} T^{a,i_1}_{j_1}\, \mathrm{i}\widetilde{\mathcal{M}}_t && (5.75)
\end{aligned}$$

Assuming n_ℓ light flavours are not distinguished in the experiment (usually $n_\ell = 4$ or $n_\ell = 5$) the squared matrix element gives

$$\overline{|\mathcal{M}_{\alpha\beta}|^2} \equiv \sum_{\gamma,\delta} \overline{|\mathcal{M}(q_\alpha \bar{q}_\beta \to q_\gamma \bar{q}_\delta)|^2} = \delta_{\alpha\beta} n_\ell \overline{|\mathcal{M}_s|^2} + \overline{|\mathcal{M}_t|^2} - 2\delta_{\alpha\beta} \mathrm{Re} \overline{\mathcal{M}_s^* \mathcal{M}_t} \qquad (5.76)$$

The initial-state flavours have to be kept fixed because every quark flavour has a different PDF.

Since the colour quantum number cannot be observed, the matrix elements are averaged over initial-state colours and summed over final-state colour, in analogy to the treatment of spin for unpolarized cross sections in (3.173). The square of the s-channel matrix element gives

$$\begin{aligned}\overline{|\mathcal{M}_s|^2} &\equiv \frac{1}{N_c}\frac{1}{N_c}\frac{1}{4}\sum_{\text{colour,spins}} |\mathcal{M}_s|^2 \\ &= \frac{1}{N_c^2} T^{a,j_2}_{j_1} T^{a,i_1}_{i_2} (T^{b,j_2}_{j_1} T^{b,i_1}_{i_2})^* \frac{1}{4}\sum_{\text{spins}} |\widetilde{\mathcal{M}}_s|^2\end{aligned} \tag{5.77}$$

Using the fact that the generators are hermitian, $(T^{a,i}_j)^* = T^{a,j}_i$ the colour factor simplifies

$$T^{a,j_2}_{j_1} T^{a,i_1}_{i_2} T^{b,j_1}_{j_2} T^{b,i_2}_{i_1} = (\operatorname{tr} T^a T^b)^2 = T_F^2 \delta_{ab}\delta_{ab} = \frac{N_c^2-1}{4} \tag{5.78}$$

The same colour factor appears in the squared t-channel matrix element. For equal quark flavours, there is also the interference term:

$$\overline{\mathcal{M}_s^* \mathcal{M}_t} = \frac{1}{N_c^2} T^{a,j_2}_{i_2} T^{a,i_1}_{j_1} T^{b,i_2}_{i_1} T^{b,j_1}_{j_2} \frac{1}{4}\sum_{\text{spins}} \widetilde{\mathcal{M}}_s^* \widetilde{\mathcal{M}}_t \tag{5.79}$$

The colour factor for this contribution can be evaluated using (4.106) and (4.92)

$$\operatorname{tr}[T^a T^b T^a T^b] = -\frac{1}{2N_c}\operatorname{tr}[T^b T^b] = -\frac{1}{2N_c}\delta_{bb} T_F = -\frac{N_c^2-1}{4N_c} = -\frac{2}{3} \tag{5.80}$$

The spin-averaged squared "colour-stripped" matrix elements $\widetilde{\mathcal{M}}_{s/t}$ are given by the same expressions as the QED matrix elements for $e^-e^+ \to \mu^-\mu^+$ and $e^-\mu^+ \to e^-\mu^+$ in (3.180) and (3.187), up to the replacement of the coupling constants, $e \to g_s$:

$$\begin{aligned}\overline{|\widetilde{\mathcal{M}}_s|^2} &= \frac{4g_s^4}{s^2} 2\left[(p_1\cdot k_1)(p_2\cdot k_2) + (p_1\cdot k_2)(k_1\cdot p_2)\right] = \frac{2g_s^4}{s^2}(t^2+u^2) \\ \overline{|\mathcal{M}_t|^2} &= \frac{4g_s^4}{t^2} 2\left[(p_1\cdot p_2)(k_1\cdot k_2) + (p_1\cdot k_2)(k_1\cdot p_2)\right] = \frac{2g_s^4}{t^2}(s^2+u^2)\end{aligned} \tag{5.81}$$

The squared interference term gives:

$$\overline{\widetilde{\mathcal{M}}_s^* \widetilde{\mathcal{M}}_t} = \frac{g_s^4}{4st}\operatorname{tr}[\not{k}_1\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_2 \gamma^\mu \not{k}_2 \gamma^\nu] = -2g_s^4 \frac{u^2}{st} \tag{5.82}$$

Using the identities

$$\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu = 4g^{\nu\rho}\,, \qquad \gamma_\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \tag{5.83}$$

the computation of the Dirac trace simplifies to

$$\begin{aligned}\operatorname{tr}[\not{k}_1\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_2 \gamma^\mu \not{k}_2 \gamma^\nu] &= -2\operatorname{tr}[\not{k}_1\gamma_\mu \not{p}_1 \not{k}_2 \gamma^\mu \not{p}_2] = -8(p_1\cdot k_2)\operatorname{tr}[\not{k}_1 \not{p}_2] \\ &= -32(p_1\cdot k_2)(k_1\cdot p_2) = -8\,u^2\end{aligned} \tag{5.84}$$

The complete result is therefore

$$\overline{|\mathcal{M}_{\alpha\beta}|^2} = 2g_s^4 \frac{N_c^2 - 1}{4N_c^2} \left\{ \frac{s^2 + u^2}{t^2} + \delta_{\alpha\beta} \left[n_\ell \left(\frac{t^2 + u^2}{s^2} \right) - \frac{2}{N_c} \frac{u^2}{st} \right] \right\} \tag{5.85}$$

The partonic cross section can be written as

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\hat{t}} = \frac{2}{\hat{s}} \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\cos\theta} = \frac{1}{\hat{s}^2} \frac{1}{8(2\pi)} \overline{|\mathcal{M}|^2} = \frac{2}{9} \frac{2\pi\alpha_s^2}{\hat{s}^2} \left\{ \frac{s^2 + u^2}{t^2} + \delta_{\alpha\beta} \left[n_\ell \left(\frac{t^2 + u^2}{s^2} \right) - \frac{2}{3} \frac{u^2}{st} \right] \right\} \tag{5.86}$$

where $\hat{t} = (p_1 - k_2)^2 = 2EE'(1 - \cos\theta) = \frac{\hat{s}}{2}(1 - \cos\theta)$.

Part II

Multi-leg Born amplitudes

Chapter 6

Spinor-helicity methods

Alternative to the computation of spin-averaged amplitudes with trace techniques: compute amplitudes for fixed helicities and perform sum over helicities explicitly. Especially useful for massless amplitudes where many helicity combinations vanish. For analytical calculations: simplifications if properties of light-like momenta are used. For this the four-vector formalism is not optimal on-shell condition not manifest. Instead: express kinematical information in terms of two-component Weyl spinors. Instead of scalar products of four-momenta, we will express the amplitudes in terms of the spinor products in (3.84)

$$\langle pk\rangle \equiv \bar{u}_L(p)u_R(k) = u_-^\dagger(p)u_+(k) \quad [pk] \equiv \bar{u}_R(p)u_L(k) = u_+^\dagger(p)u_-(k) \tag{6.1}$$

In order to exploit this method, we first need to introduce some notations.

6.1 Two-component spinors

6.1.1 Weyl Spinors

Let us discuss some properties of the spinors of massless spin 1/2 particles introduced in Section 3.2.2. Helicity eigenstates for massless Dirac fermions are given in terms of two-component spinors:

$$u_R(p) = \begin{pmatrix} u_+(p) \\ 0 \end{pmatrix} \qquad\qquad u_L(p) = \begin{pmatrix} 0 \\ u_-(p) \end{pmatrix} \tag{6.2}$$

$u_\pm$ are solutions to the Weyl equations:

$$p_\mu \sigma^\mu u_+(p) = 0, \qquad\qquad p_\mu \bar{\sigma}^\mu u_-(p) = 0 \tag{6.3}$$

Lorentz Transformations

The left-and right-handed spinors transform under Lorentz transformations in the two in-equivalent fundamental spinor representations with the transformation rules

$$D^{(0,\frac{1}{2})}: \qquad u_- \to \Lambda_L u_-\,, \qquad\qquad D^{(\frac{1}{2},0)}: \qquad u_+ \to \Lambda_R u_+ \tag{6.4}$$

where the transformations are given in terms of the angles φ_i and rapidities ν_i as

$$\Lambda_L = \exp\left(-\frac{\mathrm{i}}{2}(\vec{\varphi} - \mathrm{i}\vec{\nu})\vec{\sigma}\right), \qquad \Lambda_R = \exp\left(-\frac{\mathrm{i}}{2}(\vec{\varphi} + \mathrm{i}\vec{\nu})\vec{\sigma}\right) \tag{6.5}$$

The representations of the Lorentz transformations $\Lambda_{R,L}$ are complex 2×2 matrices with $\det \Lambda_{R,L} = 1$, i.e. $\Lambda_{R,L} \in SL(2, \mathbb{C})$.

General representations of Lorentz transformations are labeled by two half-integer values $j_{1/2}$. The Lorentz transformations in the representation $D^{(j_1, j_2)}$ are given as the matrices

$$\Lambda^{(j_1,j_2)} = \exp\left(-\mathrm{i}(\vec{\varphi} + \mathrm{i}\vec{\nu})\vec{T}_1^{(j_1)}\right) \exp\left(-\mathrm{i}(\vec{\varphi} - \mathrm{i}\vec{\nu})\vec{T}_2^{(j_2)}\right) \tag{6.6}$$

where the generators satisfy the same commutation relations as the angular momentum operators:

$$[T_a^i, T_b^j] = \mathrm{i}\epsilon^{ijk}\, T_a^k\, \delta_{ab}. \tag{6.7}$$

Right- and left-handed Weyl spinors transform in the two fundamental representations

- $D^{(\frac{1}{2},0)}$: *Right-chiral fundamental representation*

$$\text{Generators:} \qquad T_1^{(\frac{1}{2}),i} = \frac{\sigma^i}{2}, \qquad T_2^{0,i} = 0. \tag{6.8}$$

- $D^{(0,\frac{1}{2})}$: *Left-chiral fundamental representation*

$$\text{Generators:} \qquad T_1^{0,i} = 0, \qquad T_2^{(\frac{1}{2}),i} = \frac{\sigma^i}{2}. \tag{6.9}$$

6.1.2 Index notation

The relations between the left- and right-handed spinor representations can be encoded in an index notation so that the transformation properties from a spinor expression are made manifest by the types of indices. By convention, the components of left-handed spinors are denoted by upper dotted indices, the indices of right-handed spinors by undotted lower indices:

$$u_+(p) \leftrightarrow p_A \qquad u_-(p) \leftrightarrow p^{\dot{A}} \tag{6.10}$$

Here we also use the convention to use the same symbol p for the momentum and the spinor solutions of the Weyl equations for this momentum. These spinors are sometimes called **momentum spinors**.

Raising and lowering indices

Since the Lorentz transformations $\Lambda_{L/R}$ are elements of $SL(2, \mathbb{C})$, it is possible to form Lorentz invariants using the two-dimensional antisymmetric symbol ε,

$$\varepsilon^{AB} p_A k_B \to \underbrace{\det(\Lambda_R)}_{=1} \varepsilon^{AB} p_A k_B\,, \qquad \varepsilon_{\dot{A}\dot{B}} p^{\dot{A}} k^{\dot{B}} \to \underbrace{\det(\Lambda_L)}_{=1} \varepsilon_{\dot{A}\dot{B}} p^{\dot{A}} k^{\dot{B}}$$

Therefore the epsilon-symbol plays a similar role as the metric tensor in the Minkowski-product of four-vectors. For the components of the two-dimensional antisymmetric tensor we use the conventions

$$\varepsilon^{AB} = \varepsilon_{AB} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{6.11}$$

Raising and lowering of the indices of two-component Weyl spinors is then defined using the antisymmetric tensor as follows,

$$p^A = \varepsilon^{AB} p_B, \qquad\qquad p_{\dot{B}} = p^{\dot{A}} \varepsilon_{\dot{A}\dot{B}} \tag{6.12}$$

Note that

$$(\varepsilon^{-1})_{AB} = -\varepsilon_{AB} = \varepsilon_{BA} \tag{6.13}$$

so the inverse operations are defined by

$$p_B = (\varepsilon^{-1})_{BA} p^A = p^A \varepsilon_{AB} \, , \qquad\qquad p^{\dot{A}} = p_{\dot{B}} (\varepsilon^{-1})^{\dot{B}\dot{A}} = \varepsilon^{\dot{A}\dot{B}} p_{\dot{B}}, \tag{6.14}$$

In the conventions used here, dotted and undotted indices are raised and lowered in the same way. Note that indices are always contracted from "north-west" to "south-east".

Spinor products

For the Lorentz-invariant spinor products we use the following conventions:

$$\langle pk \rangle = p^A k_A = p_B k_A \varepsilon^{AB}, \tag{6.15}$$

$$[pk] = p_{\dot{A}} k^{\dot{A}} = p^{\dot{B}} \varepsilon_{\dot{B}\dot{A}} k^{\dot{A}}. \tag{6.16}$$

In the conventions used here, undotted indices in the spinor product are contracted from "north-east" to "south-west" while the dotted indices are contracted from "south-east" to "north-west". These conventions are such that the spinor products defined here are identical to the ones defined in (3.84) where we did not distinguish upper/lower and dotted indices. The antisymmetry of the spinor products implies:[1]

$$p^A k_A = -p_A k^A \qquad\qquad p^{\dot{A}} k_{\dot{A}} = -p_{\dot{A}} k^{\dot{A}} \tag{6.17}$$

Explicit expressions

From the solutions to the Weyl equations discussed in Section 3.2.2 we obtain the explicit solutions for the various spinors:

$$p_A = \frac{1}{\sqrt{(p^0 - p^3)}} \begin{pmatrix} p^1 - \mathrm{i}p^2 \\ p^0 - p^3 \end{pmatrix} \qquad p^A = \frac{1}{\sqrt{(p^0 - p^3)}} \begin{pmatrix} p^0 - p^3 \\ -p^1 + \mathrm{i}p^2 \end{pmatrix} \tag{6.18}$$

$$p_{\dot{A}} = \frac{1}{\sqrt{(p^0 - p^3)}} \begin{pmatrix} p^1 + \mathrm{i}p^2 \\ p^0 - p^3 \end{pmatrix} \qquad p^{\dot{A}} = \frac{1}{\sqrt{(p^0 - p^3)}} \begin{pmatrix} p^0 - p^3 \\ -(p^1 + \mathrm{i}p^2) \end{pmatrix} \tag{6.19}$$

[1] The rules given here apply for products of spinors whose entries are ordinary complex numbers. For spinor-valued fermion fields, the additional anti-commutation rules for fermionic operators have to be taken into account.

Conjugate spinors

The Lorentz transformations in the two fundamental representations are related by complex conjugation:

$$\begin{aligned} \varepsilon^{-1}\sigma^i\varepsilon &= -\sigma^{i*}, \\ \Rightarrow\ \Lambda_R^* &= \exp\left(\frac{\mathrm{i}}{2}(\vec{\varphi}-\mathrm{i}\vec{\nu})\vec{\sigma}^*\right) = \exp\left(-\frac{\mathrm{i}}{2}\varepsilon^{-1}(\vec{\varphi}-\mathrm{i}\vec{\nu})\vec{\sigma}\varepsilon\right) = \varepsilon^{-1}\Lambda_L\varepsilon. \end{aligned} \tag{6.20}$$

This implies the equivalence of the conjugate left (right) fundamental representation with the right (left) representation:

$$\left(D^{(\frac{1}{2},0)}\right)^* \sim D^{(0,\frac{1}{2})}, \quad \text{i.e.} \quad D^{(\frac{1}{2},0)} \xleftrightarrow[]{\substack{\text{complex}\\ \text{conjugation}}} D^{(0,\frac{1}{2})}.$$

Construction of left (right) spinors from right (left) spinors:

$$\begin{aligned} &u_+ \in D^{(\frac{1}{2},0)}: \quad (\varepsilon u_+^*) \to \varepsilon\Lambda_R^* u_+^* = \Lambda_L(\varepsilon u_+^*), \qquad \text{i.e.} \quad \varepsilon u_+^* \in D^{(0,\frac{1}{2})}, \\ &u_- \in D^{(0,\frac{1}{2})}: \quad (\varepsilon^{-1}u_-^*) \to \varepsilon^{-1}\Lambda_L^* u_-^* = \Lambda_R(\varepsilon^{-1}u_-^*), \qquad \text{i.e.} \quad (\varepsilon^{-1}u_-^*) \in D^{(\frac{1}{2},0)}. \end{aligned} \tag{6.21}$$

For momenta with real components, complex conjugation of spinors corresponds to dotting/undotting, as can be seen from the explicit formulas:

$$p_A^* = p_{\dot{A}} \qquad\qquad (p^{\dot{A}})^* = p^A \tag{6.22}$$

The relation of spinors in the various representations (6.21) reads in the index notation:

$$\begin{aligned} p^{\dot{A}} &= \varepsilon^{\dot{A}\dot{B}} p_{\dot{B}} \\ p_A &= (\varepsilon^{-1})_{AB} p^B = p^B \varepsilon_{BA} \end{aligned} \tag{6.23}$$

One sees that the relations among the different representations are "built in" the definitions of raising and lowering indices.

For the spinor products one has the relations for real momenta:

$$\langle pk\rangle^* = p^{\dot{A}} k_{\dot{A}} = [kp] \tag{6.24}$$

Dirac spinors

The components of a general Dirac spinor ψ carry indices as follows:

$$\psi = \begin{pmatrix} \chi_A \\ \bar{\xi}^{\dot{A}} \end{pmatrix}. \tag{6.25}$$

Conjugate spinor:

$$\bar{\psi} = \psi^\dagger\gamma^0 = (\xi^A, \bar{\chi}_{\dot{A}}) \tag{6.26}$$

The spinor product of two Dirac spinors breaks up into a sum of the two Weyl-spinor products

$$\bar{\psi}_1\psi_2 = \xi_1^A \chi_{2,A} + \bar{\chi}_{1,\dot{A}}\bar{\xi}_2^{\dot{A}} = \langle\xi_1\chi_2\rangle + [\bar{\chi}_1\bar{\xi}_2] \tag{6.27}$$

Schouten Identity

There is no totally antisymmetric tensor of rank three in two dimensions. This implies the **Schouten identity**:

$$\varepsilon^{AB}\varepsilon^{CD} + \varepsilon^{AC}\varepsilon^{DB} + \varepsilon^{AD}\varepsilon^{BC} = 0. \tag{6.28}$$

Contracting this expression with three spinors p_B, k_C and q_D gives the identity

$$p^A \langle kq\rangle + k^A \langle qp\rangle + q^A \langle pk\rangle = 0. \tag{6.29}$$

Writing this in the form

$$p^A = k^A \frac{\langle qp\rangle}{\langle qk\rangle} + q^A \frac{\langle kp\rangle}{\langle kq\rangle}, \tag{6.30}$$

one sees that any spinor can be decomposed in terms of two other spinors, which is the consequence of the two-dimensionality of the spinor space. Similar identities hold for dotted indices. Contracting with a fourth spinor gives the identities

$$\langle np\rangle \langle kq\rangle + \langle nk\rangle \langle qp\rangle + \langle nq\rangle \langle pk\rangle = 0\,, \qquad [np]\,[kq] + [nk]\,[qp] + [nq]\,[pk] = 0. \tag{6.31}$$

Braket notation

In the literature on helicity amplitudes also a braket notation for the spinors is often used:

$$p_A \leftrightarrow |p+\rangle = |p\rangle \qquad\qquad p^{\dot{A}} \leftrightarrow |p-\rangle = |p], \tag{6.32}$$

$$p_{\dot{A}} \leftrightarrow \langle p+| = [p| \qquad\qquad p^A \leftrightarrow \langle p-| = \langle p| \tag{6.33}$$

The spinor products are in the various equivalent notations written as

$$\langle pk\rangle = \langle p - |k+\rangle = p^A k_A \tag{6.34}$$

$$[kp] = \langle k + |p-\rangle = k_{\dot{A}} p^{\dot{A}} \tag{6.35}$$

6.2 Momenta and spinors

6.2.1 Pauli matrices

From the expression for the Dirac matrices in the chiral representation one can infer the index structure of the Pauli matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu{}_{A\dot{B}} \\ \sigma^{\mu,\dot{B}A} & \end{pmatrix} \tag{6.36}$$

Pauli Matrices are "Clebsch-Gordan coefficients" relating spinor to vector transformations:

$$\Lambda_R^\dagger \sigma^\mu \Lambda_R = \Lambda^\mu{}_\nu \sigma^\nu\,, \qquad\qquad \Lambda_L^\dagger \bar{\sigma}^\mu \Lambda_L = \Lambda^\mu{}_\nu \bar{\sigma}^\nu. \tag{6.37}$$

The Pauli matrices allow to construct 4-vectors from products of Weyl spinors:

$$u_+^\dagger(p)\sigma^\mu u_+(k) = p_{\dot A}\sigma^{\mu,\dot A B}k_B \equiv [p|\gamma^\mu|k\rangle \ , \tag{6.38}$$

$$u_-^\dagger(p)\bar\sigma^\mu u_-(k) = p^A\bar\sigma^\mu_{A\dot B}k^{\dot B} \equiv \langle p|\gamma^\mu|k] \tag{6.39}$$

Here by a slight abuse of notation the gamma matrix is used in the braket notation instead of the sigma matrices. The other combinations of spinors vanish:

$$\langle p|\gamma^\mu|k\rangle = 0 = [p|\gamma^\mu|k]$$

Raising and lowering indices

Rules for raising or lowering indices of Pauli matrices are defined consistent with the rules for lowering spinor indices:

$$\sigma^\mu_{\dot A B} = \sigma^{\mu,\dot B C}\varepsilon_{\dot B\dot A}\varepsilon_{CB} \qquad\qquad \bar\sigma^{\mu,A\dot B} = \varepsilon^{AC}\varepsilon^{\dot B\dot D}\bar\sigma_{\mu,C\dot D} \tag{6.40}$$

The property of the Pauli matrices $\varepsilon^{-1}\sigma^i\varepsilon = -\sigma^{i*}$ implies

$$\sigma^\mu_{\dot A D} = \sigma^{\mu,\dot B C}\underbrace{\varepsilon_{\dot B\dot A}}_{(\varepsilon^{-1})_{\dot A\dot B}}\varepsilon_{CD} = \underbrace{(\bar\sigma^\mu_{A\dot D})^*}_{(\bar\sigma^\mu_{A\dot D})^T} = \bar\sigma^\mu_{D\dot A} \tag{6.41}$$

where the hermiticity of the Pauli matrices was used. Similarly one finds

$$\bar\sigma^{\mu,A\dot B} = \sigma^{\mu,\dot B A} \tag{6.42}$$

Dirac algebra

The Dirac algebra $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ leads to the identities

$$\bar\sigma^\mu_{A\dot B}\sigma^{\nu,\dot B C} + \bar\sigma^\nu_{A\dot B}\sigma^{\mu,\dot B C} = 2\,\delta^C_A\,g^{\mu\nu}\,, \qquad \sigma^{\mu,\dot A B}\bar\sigma_{\nu,B\dot C} + \sigma^{\nu,\dot A B}\bar\sigma_{\mu,B\dot C} = 2\,\delta^{\dot A}_{\dot C}\,g^{\mu\nu} \tag{6.43}$$

Fierz identities

The Fierz Identity of Pauli matrices discussed in the homework reads in the index conventions introduced in this section:

$$\sigma^{\mu,\dot A B}\bar\sigma_{\mu,C\dot D} = 2\delta^{\dot A}_{\dot D}\delta^B_C \tag{6.44}$$

Raising or lowering indices leads to the identities

$$\sigma^\mu_{\dot A A}\bar\sigma_{\mu,B\dot B} = 2\varepsilon_{AB}\varepsilon_{\dot A\dot B} \quad , \quad \sigma^{\mu,\dot A A}\bar\sigma^{B\dot B}_\mu = 2\varepsilon^{AB}\varepsilon^{\dot A\dot B} \tag{6.45}$$

The Fierz identity allows to invert the mapping of spinors to vectors in (6.39)

$$\begin{aligned}\langle p|\gamma^\mu|k]\sigma_{\mu,B\dot B} &= (p^A\bar\sigma^\mu_{A\dot A}k^{\dot A})\,\sigma_{\mu,B\dot B} = 2(p^A\varepsilon_{AB})\,(k^{\dot A}\varepsilon_{\dot A\dot B}) = 2p_Bk_{\dot B}\\ [p|\gamma^\mu|k\rangle\,\bar\sigma^{B\dot B}_\mu &= (p_{\dot A}\sigma^{\mu\dot A A}k_A)\,\bar\sigma^{B\dot B}_\mu = 2(p_{\dot A}\varepsilon^{\dot A\dot B})\,(k_A\varepsilon^{AB}) = 2p^{\dot B}k^B\end{aligned} \tag{6.46}$$

Contracting these identities with two more spinors we obtain the relations

$$\langle p|\gamma^\mu|k]\,\langle q|\gamma_\mu|l] = 2\,\langle pq\rangle\,[lk]\ , \qquad\qquad [p|\gamma^\mu|k\rangle\,[q|\gamma_\mu|l\rangle = 2\,[pq]\,\langle lk\rangle \tag{6.47}$$

6.2.2 Four-momenta

Factorization of momenta into spinors

The contraction of a four-momentum p^μ with the Pauli-matrices defines the matrices

$$p_\mu \sigma^{\mu \dot{A}B} \equiv p^{\dot{A}B} \qquad\qquad p_\mu \bar{\sigma}^\mu_{A\dot{B}} \equiv p_{A\dot{B}} \tag{6.48}$$

The completeness relation of the Dirac spinors

$$\sum_{\lambda=L/R} u_\lambda(p)\bar{u}_\lambda(p) = \not{p} = \begin{pmatrix} 0 & p_\mu\bar{\sigma}^\mu \\ p_\mu\sigma^\mu & 0 \end{pmatrix} \tag{6.49}$$

implies the relations for the Weyl spinors:

$$u_+(p)\, u_+^\dagger(p) = \bar{\sigma}^\mu p_\mu, \qquad\qquad u_-(p)\, u_-^\dagger(p) = \sigma^\mu p_\mu \tag{6.50}$$

In the index notation, these identities become

$$p_{A\dot{B}} = p_A p_{\dot{B}} \qquad\qquad p^{\dot{A}B} = p^{\dot{A}} p^B \tag{6.51}$$

Because of this factorization one sometimes loosely refers to the momentum spinors as the "square roots" of the four-momentum.

In the braket notation, the identities (6.50) can be written as

$$\bar{\sigma}^\mu p_\mu = |p+\rangle\,\langle p+| = |p\rangle\,[p| \qquad\qquad \sigma^\mu p = |p-\rangle\,\langle p-| = |p]\,\langle p| \tag{6.52}$$

The factorization (6.51) can be interpreted as a decomposition of the matrices

$$p_\mu\sigma^\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + \mathrm{i}p^2 \\ -p^1 - \mathrm{i}p^2 & p^0 + p^3 \end{pmatrix}, \qquad\qquad p_\mu\bar{\sigma}^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - \mathrm{i}p^2 \\ p^1 + \mathrm{i}p^2 & p^0 - p^3 \end{pmatrix} \tag{6.53}$$

into eigenvectors. Note that

$$\det(p_\mu\sigma^\mu) = \det(p_\mu\bar{\sigma}^\mu) = p^2 = 0 \tag{6.54}$$

so that one eigenvalue vanishes and only one eigenvector exists.

Constructing momenta from spinors

The inverse identity that allows to construct the four-momentum from the spinors follows from the normalization of the Dirac spinors,

$$\bar{u}_{L/R}(p)\gamma^\mu u_{L/R}(p) = 2p^\mu. \tag{6.55}$$

In the index notation, this becomes

$$p_{\dot{A}}\sigma^{\mu,\dot{A}B} p_B = 2p^\mu\,, \qquad\qquad p^A \bar{\sigma}^\mu_{A\dot{B}} p^{\dot{B}} = 2p^\mu \tag{6.56}$$

Scaling

Note that the spinors associated to a momentum p^μ are not completely fixed since the relations (6.51) and (6.56) are invariant under a rescaling

$$p^A \to z\, p^A\,, \qquad\qquad p^{\dot{A}} \to z^{-1}\, p^{\dot{A}} \tag{6.57}$$

For real momenta, the condition $(p^A)^* = p^{\dot{A}}$ reduces the complex number z to a phase, $z = e^{i\phi}$.

Since the transformation (6.57) leaves the momentum invariant, it is an element of the **little group** of the momentum discussed in 3.1.2.

6.2.3 Gluon polarization vectors

The usefulness of the spinor notation for amplitudes in QCD comes from the fact that gluon polarization vectors can be expressed in terms of Weyl spinors.

The polarization vectors are only determined up to gauge transformations $\epsilon^\mu \to \epsilon^\mu + \alpha k^\mu$. In the spinor-helicity formalism this is reflected by introducing arbitrary so-called reference spinors $|q\rangle$ and $|q]$. The polarization vectors can then be defined as

$$\epsilon^\mu_-(k,q) = \epsilon^{\mu *}_+(k,q) = \frac{\langle q|\,\gamma^\mu |k]}{\sqrt{2}\,\langle qk\rangle}\,, \quad \epsilon^\mu_+(k,q) = \epsilon^{\mu *}_-(k,q) = \frac{[q|\gamma^\mu\,|k\rangle}{\sqrt{2}\,[kq]} \tag{6.58}$$

The polarization vectors can be also given in spinor components as

$$\epsilon^*_{+,A\dot{B}}(k,q) = \sqrt{2}\frac{q_A k_{\dot{B}}}{\langle qk\rangle} \qquad\qquad \epsilon^*_{-,A\dot{B}}(k,q) = \sqrt{2}\frac{k_A q_{\dot{B}}}{[kq]} \tag{6.59}$$

These are correct expressions for polarization vectors since they satisfy the properties ($\Rightarrow$ homework):

- Transversality:
$$k_\mu \epsilon^\mu_\pm(k,q) = 0 = q_\mu \epsilon^\mu_\pm(k,q) \tag{6.60}$$
- Normalization:
$$\epsilon_\lambda(k,q)\cdot\epsilon^*_{\lambda'}(k,q) = -\delta_{\lambda,\lambda'} \tag{6.61}$$
- Completeness relation:
$$\sum_{\lambda=\pm} \epsilon^\mu_\lambda(k,q)\epsilon^{\nu *}_\lambda(k,q) = -g^{\mu\nu} + \frac{k^\mu q^\nu + q^\mu k^\nu}{(k\cdot q)}$$
- Gauge transformations: a change of the reference spinors corresponds to a shift of the polarization vector of the form $\epsilon \to \epsilon + \alpha k$:
$$\epsilon^{\mu *}_+(k,q) - \epsilon^{\mu *}_+(k,q') = \sqrt{2}\frac{\langle q'q\rangle}{\langle q'k\rangle\,\langle qk\rangle}\, k^\mu \tag{6.62}$$

Note that the polarization vectors transform under the scaling of the momentum spinors (6.57) as

$$\epsilon_-^\mu \to z^{-2}\epsilon_-^\mu\,, \qquad \epsilon_+^\mu \to z^2\epsilon_+^\mu \tag{6.63}$$

The consideration of such "little group scalings" will allow to constrain the possible interaction vertices of massless particles.

6.2.4 Rules for calculations with Weyl spinors

Properties of spinor products

- Relation of Minkowski product and spinor products: Contracting (6.55) with a second light-like four-momentum allows to express Minkowski products of light-like four momenta in terms of spinor products:

  $$2\,(p\cdot k) = [p|\not{k}\,|p\rangle = p_{\dot{A}}k^{\nu,\dot{A}B}p_B = [pk]\,\langle kp\rangle \tag{6.64}$$

  Alternatively, this identity can be derived from the Dirac algebra (6.43).

- Symmetry of matrix element of Pauli matrices ($\Rightarrow$ homework):

  $$\langle p+|\gamma^\mu|k+\rangle = \langle k-|\gamma^\mu|p-\rangle \tag{6.65}$$

  Generalizations:

  $$\langle p\pm|\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}|q\pm\rangle = \langle q\mp|\gamma^{\mu_{2n+1}}\dots\gamma^{\mu_1}|p\mp\rangle \tag{6.66}$$
  $$\langle p\pm|\gamma^{\mu_1}\dots\gamma^{\mu_{2n}}|q\mp\rangle = -\,\langle q\pm|\gamma^{\mu_{2n}}\dots\gamma^{\mu_1}|p\mp\rangle \tag{6.67}$$

- For real momenta, complex conjugation gives

  $$\langle p+|\gamma^\mu|k+\rangle^* = (p_{\dot{A}}\sigma^{\dot{A}B}k_B)^* = (p_A\underbrace{(\sigma^{\dot{A}B})^*}_{=\bar{\sigma}^{A\dot{B}}}k_{\dot{B}}) = \langle p-|\gamma^\mu|k-\rangle \tag{6.68}$$

Momentum conservation

Consider a scattering process with momentum conservation $p_1+p_2 = k_1+\dots k_n$. If all particles are massless, momentum conservation implies the identity

$$\langle ap_1\rangle\,[p_1b] + \langle ap_2\rangle\,[p_2b] = \sum_i \langle ak_i\rangle\,[k_ib] \tag{6.69}$$

for arbitrary spinors $\langle a-|$ and $|b-\rangle$. Choosing these spinors in terms of external momenta, one can obtain simpler identities, e.g.

$$0 = \langle p_1-|\not{k}_1+\dots+\not{k}_n|p_2-\rangle = \sum_i \langle p_1k_i\rangle\,[k_ip_2] \tag{6.70}$$

Here it was used that

$$\ldots \not{p}_2 \, |p_2 \pm\rangle = 0 \tag{6.71}$$

because of the Weyl equation. In particular, for a four-particle scattering process one finds simple identities such as

$$\langle p_1 k_1\rangle \, [k_1 p_2] = -\, \langle p_1 k_2\rangle \, [k_2 p_2] \tag{6.72}$$

External states

For the application of the spinor-helicity method to the calculation of scattering amplitudes we collect here the appropriate external spinors or polarization vectors.

Incoming particles:

$$\begin{aligned}
\text{quarks}: \quad & q_L(p) : u_L(p) \to p^{\dot{A}} \to |p-\rangle = |p] \\
& q_R(p) : u_R(p) \to p_A \to |p+\rangle = |p\rangle \qquad (6.73) \\
\text{antiquarks}: \quad & \bar{q}_L(p) : \bar{v}_R(p) \to p_{\dot{A}} \to \langle p+| = [p| \\
& \bar{q}_R(p) : \bar{v}_L(p) \to p^A \to \langle p-| = \langle p| \qquad (6.74) \\
\text{gluons}: \quad & g_-(p) : \epsilon^\mu_-(p,q) = \frac{\langle q|\, \gamma^\mu |k]}{\sqrt{2}\,\langle qk\rangle} \\
& g_+(p) : \epsilon^\mu_+(p,q) = \frac{[q|\gamma^\mu\, |k\rangle}{\sqrt{2}\,[kq]} \qquad (6.75)
\end{aligned}$$

Outgoing particles:

$$\begin{aligned}
\text{quarks}: \quad & q_L(k) : \bar{u}_L(k) \to k^A \to \langle k-| = \langle k| \\
& q_R(k) : \bar{u}_R(k) \to k_{\dot{A}} \to \langle k+| = [k| \qquad (6.76) \\
\text{antiquarks}: \quad & \bar{q}_L(k) : v_R(k) \to k_A \to |k+\rangle = |k\rangle \\
& \bar{q}_R(k) : v_L(k) \to k^{\dot{A}} \to |k-\rangle = |k] \qquad (6.77) \\
\text{gluons}: \quad & g_-(p) : \epsilon^{\mu,*}_-(p,q) = \frac{[q|\gamma^\mu\, |k\rangle}{\sqrt{2}\,[kq]} \\
& g_+(p) : \epsilon^{\mu,*}_+(p,q) = \frac{\langle q|\, \gamma^\mu |k]}{\sqrt{2}\,\langle qk\rangle} \qquad (6.78)
\end{aligned}$$

One sees that the conventions are such that for **outgoing** states the momentum spinors with angular brackets correspond to left-handed states and spinors with square brackets correspond to right-handed states while for incoming states the relation is reversed.

6.3 Examples

As first examples for the application of the spinor-helicity method, we consider the process $e^- e^+ \to q\bar{q}$ in QED, and the process with an additional gluon, $e^- e^+ \to q\bar{q}g$.

6.3.1 $e^-e^+ \to q\bar{q}$

The matrix element is given by

$$\begin{aligned}
\mathrm{i}\mathcal{M}(e^{-,\lambda_1}_{p_1} e^{+,\lambda_2}_{p_2} \to q^{\sigma_1}_{k_1} \bar{q}^{\sigma_2}_{k_2}) &= (-\mathrm{i}e)^2 \langle q^{\sigma_1}_{k_1} \bar{q}^{\sigma_2}_{k_2} | j_{q,\mu}(0)|0\rangle \frac{-\mathrm{i}g^{\mu\nu}}{(p_1+p_2)^2+\mathrm{i}\epsilon}(\bar{v}_{\lambda_2}(p_2)\gamma_\nu u_{\lambda_1}(p_1)) \\
&= (-\mathrm{i}e)^2 Q_q (\bar{u}_{\sigma_1}(k_1)\gamma_\mu v_{\sigma_2}(k_2)) \frac{-\mathrm{i}g^{\mu\nu}}{(p_1+p_2)^2+\mathrm{i}\epsilon}(\bar{v}_{\lambda_2}(p_2)\gamma_\nu u_{\lambda_1}(p_1)).
\end{aligned} \tag{6.79}$$

The only non-vanishing helicity combinations are those where different helicities enter the same vertex. As a first example consider

$$\begin{aligned}
\mathrm{i}\mathcal{M}(e^{-,L}_{p_1} e^{+,R}_{p_2} \to q^{L}_{k_1} \bar{q}^{R}_{k_2}) &= (-\mathrm{i}e)^2 (\bar{u}_L(k_1)\gamma_\mu v_L(k_2)) \frac{-\mathrm{i}}{(p_1+p_2)^2+\mathrm{i}\epsilon}(\bar{v}_L(p_2)\gamma^\mu u_L(p_1)) \\
&= (-\mathrm{i}e)^2 \langle k_1|\,\gamma_\mu|k_2] \frac{-\mathrm{i}}{(p_1+p_2)^2} \langle p_2|\,\gamma^\mu|p_1] \\
&= \mathrm{i}e^2 \frac{2\langle k_1p_2\rangle\,[p_1k_2]}{\langle p_1p_2\rangle\,[p_2p_1]} = 2\mathrm{i}e^2 \frac{\langle k_1p_2\rangle^2}{\langle p_1p_2\rangle\,\langle k_2k_1\rangle}
\end{aligned} \tag{6.80}$$

where the matrix elements of the Dirac matrices have been combined using the Fierz identity (6.47),

$$\langle k_1|\,\gamma_\mu|k_2]\,\langle p_2|\,\gamma^\mu|p_1] = 2\,\langle k_1p_2\rangle\,[p_1k_2] \tag{6.81}$$

and where we have used momentum conservation in the form

$$[p_1p_2]\,\langle p_2k_1\rangle = [p_1k_2]\,\langle k_2k_1\rangle \tag{6.82}$$

The squared amplitude can be computed using identities such as

$$\langle k_1p_2\rangle\,\langle k_1p_2\rangle^* = \langle k_1p_2\rangle\,[p_2k_1] = 2(p_2\cdot k_1) \tag{6.83}$$

This gives the expression

$$|\mathcal{M}(e^{L^-}_{p_1} e^{+,R}_{p_2} \to q^{L}_{k_1} \bar{q}^{R}_{k_2})|^2 = 4e^4 \frac{(p_2\cdot k_1)^2}{(p_1\cdot p_2)(k_1\cdot k_2)} = 4e^4\frac{u^2}{s^2} \tag{6.84}$$

where the Mandelstam variable $u = (p_2-k_1)^2 = 2(p_2\cdot k_1)$ was used.

The amplitude for the second helicity combination is obtained by exchanging the final-state helicities:

$$\begin{aligned}
\mathrm{i}\mathcal{M}(e^{-,L}_{p_1} e^{+,R}_{p_2} \to q^{R}_{k_1} \bar{q}^{L}_{k_2}) &= (-\mathrm{i}e)^2 [k_1|\gamma_\mu\,|k_2\rangle \frac{-\mathrm{i}}{(p_1+p_2)^2} \langle p_2|\,\gamma^\mu|p_1] \\
&= \mathrm{i}e^2 \frac{2\,[k_1p_1]\,\langle p_2k_2\rangle}{\langle p_1p_2\rangle\,[p_2p_1]} = 2\mathrm{i}e^2 \frac{\langle p_2k_2\rangle^2}{\langle p_1p_2\rangle\,\langle k_2k_1\rangle}
\end{aligned} \tag{6.85}$$

using

$$[p_1p_2]\,\langle p_2k_2\rangle = [p_1k_1]\,\langle k_1k_2\rangle \tag{6.86}$$

The squared amplitude gives

$$|\widetilde{\mathcal{M}}_s(e^{-,L}_{p_1}\bar{e}^{+,R}_{p_2} \to q^R_{k_1}\bar{q}^L_{k_2})|^2 = 4e^4\frac{(p_2\cdot k_2)^2}{(p_1\cdot p_2)(k_1\cdot k_2)} = 4e^4\frac{t^2}{s^2} \tag{6.87}$$

The amplitudes with all helicities flipped are obtained by turning angular brackets into square brackets, so the squared amplitudes are the same. Summing over final-state helicities and averaging over initial-state helicities gives

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{1}{4}\left(2|\mathcal{M}(e^{-,L}_{p_1}e^{+,R}_{p_2} \to q^L_{k_1}\bar{q}^R_{k_2})|^2 + 2|\mathcal{M}(e^{-,L}_{p_1}e^{+,R}_{p_2} \to q^R_{k_1}\bar{q}^L_{k_2})|^2\right)\\ &= 2\frac{e^4}{s^2}(u^2+t^2)\end{aligned} \tag{6.88}$$

Therefore we reproduce the result (5.81) obtained using the usual approach to compute the spin-averaged matrix element.

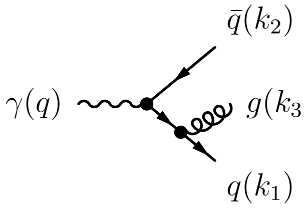
6.3.2 $e^-e^+ \to q\bar{q}g$

As a first example for the application of the spinor-helicity method to external gauge bosons, we add a real emitted gluon in the final state to the process $e^-e^+ \to q\bar{q}$. Using the notation introduced in the discussion of $e^-e^+ \to$ hadrons in Section 5.1, the matrix element for the process $e^-e^+ \to q\bar{q}g$ can be expressed in terms of the leptonic spinor chain for the subprocess $e^-e^+ \to \gamma$ and the expectation value of the quark current j^μ_q:

$$\mathrm{i}\mathcal{M}^\mu(e^{-,\lambda_1}_{p_1}e^{+,\lambda_2}_{p_2} \to q^{\sigma_1}_{k_1}\bar{q}^{\sigma_2}_{k_2}g^{\sigma_3}_{k_3}) = (-\mathrm{i}e)^2\,\langle q^{\sigma_1}_{k_1}\bar{q}^{\sigma_2}_{k_2}g^{\sigma_3}_{k_3}|j_{q,\mu}(0)|0\rangle\,\frac{-\mathrm{i}g^{\mu\nu}}{q^2+\mathrm{i}\epsilon}(\bar{v}_{\lambda_2}(p_2)\gamma_\nu u_{\lambda_1}(p_1)). \tag{6.89}$$

At tree level, the expectation value of the quark current is

$\langle q^{\sigma_1}_{k_1}\bar{q}^{\sigma_2}_{k_2}g^{\sigma_3}_{k_3}|j_{q,\mu}(0)|0\rangle =$



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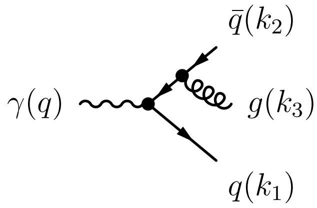


$$\begin{aligned}&=(-\mathrm{i}g_s)T^{a,i_1}_{i_2}Q_q\\ &\quad\bar{u}_{\sigma_1}(k_1)\left[\not{\epsilon}^*_{\sigma_3}(k_3)\frac{\mathrm{i}(\not{k}_1+\not{k}_3)}{(k_1+k_3)^2}\gamma^\mu + \gamma^\mu\frac{\mathrm{i}(\not{k}_2+\not{k}_3)}{(k_2+k_3)^2}\not{\epsilon}^*_{\sigma_3}(k_3)\right]v_{\sigma_2}(k_2)\\ &\equiv Q_q g_s T^{a,i_1}_{i_2}J^\mu(q^{\sigma_1}_{k_1}\bar{q}^{\sigma_2}_{k_2}g^{\sigma_3}_{k_3})\end{aligned} \tag{6.90}$$

Considering as an example the helicity combination $q^R\bar{q}^Lg^+$, the quark current becomes:

$$J^\mu(q^R_{k_1}\bar{q}^L_{k_2}g^+_{k_3}) = [k_1|\left[\frac{\not{\epsilon}^*_+(k_3)(\not{k}_1+\not{k}_3)\gamma^\mu}{(k_1+k_3)^2} + \frac{\gamma^\mu(\not{k}_2+\not{k}_3)\not{\epsilon}^*_+(k_3)}{(k_2+k_3)^2}\right]|k_2\rangle \tag{6.91}$$

Inserting the explicit form of the polarization vector we can use the Fierz identities to write

$$\not{\epsilon}^*_+(k_3, q) = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \frac{\langle q| \gamma^\mu |k_3]}{\sqrt{2}\langle qk_3\rangle} = \frac{\sqrt{2}}{\langle qk_3\rangle} \begin{pmatrix} 0 & |k_3]\langle q| \\ |q\rangle [k_3| & 0 \end{pmatrix} \tag{6.92}$$

Therefore the matrix element of the current can be written as

$$\begin{aligned} J^\mu(q^L_{k_1}\bar{q}^R_{k_2}g^-_{k_3}) &= \frac{\sqrt{2}}{\langle qk_3\rangle}\left[\frac{[k_1k_3]\langle q|(\not{k}_1+\not{k}_3)\gamma^\mu|k_2\rangle}{\langle k_1k_3\rangle [k_3k_1]} + \frac{[k_1|\gamma^\mu \not{k}_2|k_3]\langle qk_2\rangle}{\langle k_2k_3\rangle [k_3k_2]}\right] \\ &= \frac{\sqrt{2}}{\langle qk_3\rangle}\left[\frac{\langle q|(\not{k}_1+\not{k}_3)\gamma^\mu|k_2\rangle}{\langle k_3k_1\rangle} + \frac{[k_1|\gamma^\mu|k_2\rangle\langle qk_2\rangle}{\langle k_3k_2\rangle}\right] \end{aligned} \tag{6.93}$$

At this point we can simplify the calculation by making a choice for the arbitrary reference spinor $|q\rangle$. Setting $|q\rangle = |k_2\rangle$ eliminates the second term and one gets

$$J^\mu(q^R_{k_1}\bar{q}^L_{k_2}g^+_{k_3}) = \sqrt{2}\frac{\langle k_2|(\not{k}_1+\not{k}_3)\gamma^\mu|k_2\rangle}{\langle k_2k_3\rangle\langle k_3k_1\rangle} \tag{6.94}$$

The complete amplitude can be simplified using momentum conservation and the Fierz identity:

$$\begin{aligned} \mathrm{i}\mathcal{M}(e^{-,L}_{p_1}e^{+,R}_{p_2} \to q^R_{k_1}\bar{q}^L_{k_2}g^+_{k_3}) &= (-\mathrm{i}e)^2 \langle q^{\sigma_1}_{k_1}\bar{q}^{\sigma_2}_{k_2}g^{\sigma_3}_{k_3}|j_{q,\mu}(0)|0\rangle \frac{-\mathrm{i}g^{\mu\nu}}{(p_1+p_2)^2}(\bar{v}_L(p_2)\gamma_\nu u_L(p_1)) \\ &= -\mathrm{i}e^2 Q_q g_s T^{a,i_1}_{i_2}\sqrt{2}\frac{\langle k_2|(\not{p}_1+\not{p}_2)\gamma_\mu|k_2\rangle}{\langle p_1p_2\rangle [p_2p_1]\langle k_2k_3\rangle\langle k_3k_1\rangle}\langle p_2|\gamma^\mu|p_1] \\ &= -\mathrm{i}e^2 Q_q g_s T^{a,i_1}_{i_2}\sqrt{2}\frac{2\langle k_2|\not{p}_2|p_1]\langle p_2k_2\rangle}{\langle p_1p_2\rangle [p_2p_1]\langle k_2k_3\rangle\langle k_3k_1\rangle} \\ &= \mathrm{i}e^2 Q_q g_s T^{a,i_1}_{i_2}\sqrt{2}\frac{2\langle p_2k_2\rangle^2}{\langle p_1p_2\rangle\langle k_2k_3\rangle\langle k_3k_1\rangle} \end{aligned} \tag{6.95}$$

Note the similarity of this result to the corresponding amplitude without the additional gluon (6.85).

For the amplitude with a negative helicity gluon one can choose $|q] = |k_1]$ and one obtains

$$\begin{aligned} \mathrm{i}\mathcal{M}(e^{-,L}_{p_1}e^{+,R}_{p_2} \to q^R_{k_1}\bar{q}^L_{k_2}g^-_{k_3}) &= -\mathrm{i}e^2 Q_q g_s T^{a,i_1}_{i_2}\sqrt{2}\frac{[k_1|(\not{p}_1+\not{p}_2)\gamma_\mu|k_1]}{\langle p_1p_2\rangle [p_2p_1][k_2k_3][k_3k_1]}\langle p_2|\gamma^\mu|p_1] \\ &= -\mathrm{i}e^2 Q_q g_s T^{a,i_1}_{i_2}\sqrt{2}\frac{2[k_1|\not{p}_1|p_2\rangle [p_1k_1]}{\langle p_1p_2\rangle [p_2p_1][k_2k_3][k_3k_1]} \\ &= \mathrm{i}e^2 Q_q g_s T^{a,i_1}_{i_2}\sqrt{2}\frac{2[p_1k_1]^2}{[p_1p_2][k_2k_3][k_3k_1]} \end{aligned} \tag{6.96}$$

Chapter 7

Colour decomposition

Before we apply the spinor-helicity method to genuine QCD process, it is useful to discuss the colour structure of QCD amplitudes in more detail.

- QCD Feynman rules are a combination of kinematic factors and the colour structures T^a, f^{abc}, $f^{abe}f^{cde}$.

- Idea of **colour decomposition**: separate colour structure from kinematic structure.

- Colour decompositions are not unique, for a discussion of different possibilities see Section 6 of [10].

- Here we use the approach to express all colour structures in terms of generators T^a using

$$f^{abc} = -2\mathrm{i}\,\mathrm{tr}[[T^a, T^b]T^c] \tag{7.1}$$

- The resulting traces can be combined using colour Fierz identity (4.104)

$$T^{a,i}{}_j T^{a,k}{}_l = \frac{1}{2}\left(\delta^i{}_l \delta^k{}_j - \frac{1}{N_c}\delta^i{}_j \delta^k{}_l\right) \tag{7.2}$$

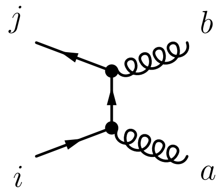
- In the following the momenta of all gluons will be considered to be outgoing to treat all gluons on the same footing, physical amplitudes with initial state gluons can be obtained by crossing.

7.1 Examples

7.1.1 $q\bar{q} \to gg$

The three diagrams contributing to the process $q\bar{q} \to gg$ have the colour structures:

$\mathrm{i}\mathcal{M} =$



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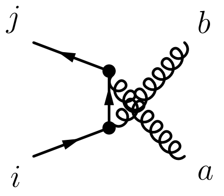

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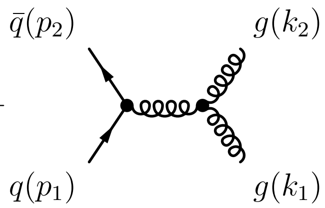


(7.3)

$(T^bT^a)^i_j \qquad (T^aT^b)^i_j \qquad f^{abc}T^{c,i}_j$

The three-gluon vertex can be split into a colour-part and a kinematic part (note that all momenta here are outgoing):

$$V^{\mu_1\mu_2\mu_3}_{abc}(p_1,p_2,p_3) = g_s f^{abc} \underbrace{\left[g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3} + g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}\right]}_{\equiv V^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)} \quad (7.4)$$

The colour-structure of the s-channel diagram can be simplified using the Lie Algebra:

$$\begin{aligned} f^{abc}T^{c,i}_j V^{\mu_1\mu_2\mu_3}(k_1,k_2,q) &= (-\mathrm{i})[T^a,T^b]^i_j V^{\mu_1\mu_2\mu_3}(k_1,k_2,q) \\ &= \mathrm{i}(T^bT^a)^i_j V^{\mu_1\mu_2\mu_3}(k_1,k_2,q) + \mathrm{i}(T^aT^b)^i_j V^{\mu_2\mu_1\mu_3}(k_2,k_1,q) \end{aligned} \quad (7.5)$$

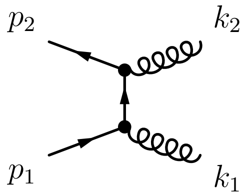
$\Rightarrow$ Two colour structures contribute to amplitude:

$$\begin{aligned} \mathcal{M}(q_{i,p_1}\bar{q}_{j,p_2} \to g_{a,k_1}g_{b,k_2}) =& g_s^2(T^bT^a)^i_j M(q_{p_1}\bar{q}_{p_2} \to g_{k_1}g_{k_2}) \\ &+ g_s^2(T^aT^b)^i_j M(q_{p_1}\bar{q}_{p_2} \to g_{k_2}g_{k_1}) \end{aligned} \quad (7.6)$$

with so-called **partial amplitudes** M. In this example, we have the following properties of the partial amplitudes:

- Only two diagrams contribute to the partial amplitudes:

$\mathrm{i}M =$



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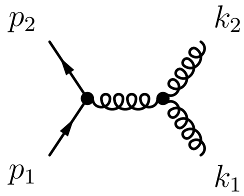
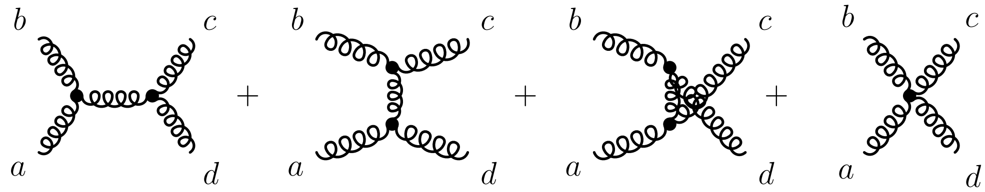


- Fixed ordering of external legs (the u-channel diagram with exchanged ordering of the gluons does not contribute) $\Rightarrow$ "colour ordered" amplitudes.
- The two partial amplitudes are related by exchanging $k_1 \leftrightarrow k_2$.
- The two colour structures are linearly independent $\Rightarrow$ the partial amplitudes are individually gauge invariant.

7.1.2 $gg \to gg$

There are four diagrams with three different colour structures for the four-gluon amplitude:

$\mathrm{i}\mathcal{M} =$



(7.7)

$$f^{abe} f^{cde} \qquad f^{ade} f^{bce} \qquad f^{ace} f^{bde} \qquad \begin{pmatrix} f^{ade} f^{bce} \\ f^{ace} f^{bde} \\ f^{ade} f^{bce} \end{pmatrix}$$

Using the Fierz identity (4.104) the colour structures can be written as

$$\begin{aligned} f^{abe} f^{cde} &= (-2\mathrm{i})^2 \,\mathrm{tr}[[T^a, T^b]T^e] \,\mathrm{tr}[[T^c, T^d]T^e] \\ &= (-2\mathrm{i})^2 [T^a, T^b]^i_j \, [T^c, T^d]^k_l \, T^{e,j}_i T^{e,l}_k \\ &= -2 \,\mathrm{tr}[[T^a, T^b][T^c, T^d]] \end{aligned} \tag{7.8}$$

The colour-suppressed $1/N_c$ terms vanish since

$$\mathrm{tr}[[T^a, T^b]] \sim f^{abc} \,\mathrm{tr}\, T^c = 0. \tag{7.9}$$

Treating the remaining three diagrams in the same way results in six independent colour structures of the form

$$T(abcd) = \mathrm{Tr}\left(T^a T^b T^c T^d\right) \tag{7.10}$$

that are not related by cyclic permutations:

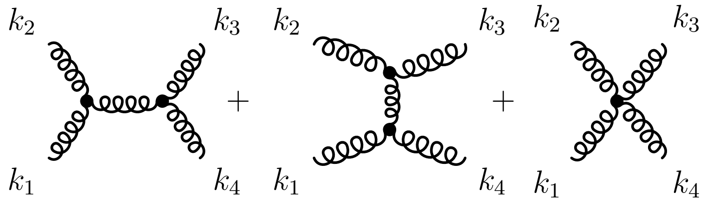
$$T(abcd), \quad T(acdb), \quad T(abdc), \quad T(adcb), \quad T(adbc), \quad T(acbd) \tag{7.11}$$

The amplitude can therefore be decomposed into 6 partial amplitudes:

$$\begin{aligned} \mathcal{M}_n(g_{a,k_1}, g_{b,k_2}, g_{c,k_3}, g_{d,k_4}) =& 2g_s^2 \,\mathrm{Tr}\left(T^a T^b T^c T^d\right) M\left(g_{k_1}, g_{k_2}, g_{k_3}, g_{k_4}\right) \\ &+ \text{non-cyclic permutations} \end{aligned} \tag{7.12}$$

The factor 2 is extracted by convention. It can be seen that only three diagrams contribute to each partial amplitude, where the external gluons are in a fixed order:

$\mathrm{i}M(g_{k_1}, g_{k_2}, g_{k_3}, g_{k_4}) =$



(7.13)

7.2 Colour ordered Feynman rules

The colour ordered diagrams can be computed directly using modified "colour-less" Feynman rules. We first write the three and four-gluon vertices in a form where each term corresponds to a fixed ordering and then give the appropriate Feynman rules.

7.2.1 Colour ordered vertices

Three gluon vertex

$$\begin{aligned} V_{abc}^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) =& g_s \underbrace{f^{abc}}_{=-2\mathrm{i}\,\mathrm{tr}[[T^a,T^b]T^c]} V^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) \\ =& -2g_s\mathrm{i}\left(\mathrm{tr}[T^aT^bT^c]V^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)+\mathrm{tr}[T^aT^cT^b]V^{\mu_1\mu_3\mu_2}(p_1,p_3,p_2)\right) \end{aligned} \tag{7.14}$$

$\Rightarrow$ two terms with fixed order of gluons.

Four gluon vertex

$$\begin{aligned} V_{abcd}^{\mu_1\mu_2\mu_3\mu_4} = 2\mathrm{i}g_s^2\Big[& \mathrm{tr}[[T^a,T^b][T^c,T^d]](g^{\mu_1\mu_3}g^{\mu_2\mu_4}-g^{\mu_2\mu_3}g^{\mu_1\mu_4}) \\ & + \mathrm{tr}[[T^a,T^c][T^b,T^d]](g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_2\mu_3}g^{\mu_1\mu_4}) \\ & + \mathrm{tr}[[T^a,T^d][T^b,T^c]](g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_4\mu_2}g^{\mu_3\mu_1})\Big] \\ & = 2\mathrm{i}g_s^2\,\mathrm{tr}[T^aT^bT^cT^d]\left(g^{\mu_1\mu_3}g^{\mu_2\mu_4}-g^{\mu_2\mu_3}g^{\mu_1\mu_4}\right)-\left(g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_4\mu_2}g^{\mu_3\mu_1}\right)] \\ & + \text{non-cyclic colour structures} \\ & = 2\mathrm{i}g_s^2\,\mathrm{tr}[T^aT^bT^cT^d](2g^{\mu_1\mu_3}g^{\mu_2\mu_4}-g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_2\mu_3}g^{\mu_1\mu_4})+\ldots \end{aligned} \tag{7.15}$$

$\Rightarrow$ 6 colour structures with fixed ordering as discussed above for the amplitude.

7.2.2 Colour ordered multi-gluon amplitudes

The colour structure found for the four-point amplitudes generalizes to amplitudes with a quark anti-quark pair and an arbitrary number of gluons:

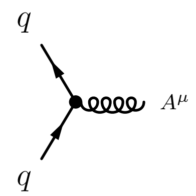
$$\begin{aligned} &\mathcal{M}_n(\bar{q}_1^i,q_{2,j},g_3,g_4,...,g_n) = \\ &\quad g_s^{n-2}\sum_{\sigma\in S_{n-2}(3,\ldots,n)}\left(T^{a_{\sigma(3)}}...T^{a_{\sigma(n)}}\right)_j^i M_n\left(\bar{q}_1,q_2,g_{\sigma(3)},...,g_{\sigma(n)}\right), \end{aligned} \tag{7.16}$$

where the M_n are n-point color ordered partial amplitudes and the sum is over all permutations of the external gluon legs. Here all particles are treated as outgoing, therefore the role of upper and lower indices for quark and antiquark is reversed.

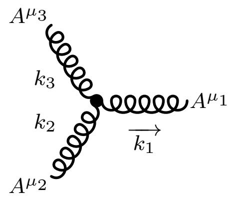
In the pure gluonic case tree level amplitudes with n external gluons may be written in the form

$$\mathcal{M}_n(g_1, g_2, ..., g_n) = g_s^{n-2} \sum_{\sigma} 2\ \mathrm{Tr}\left(T^{a_{\sigma(1)}} ... T^{a_{\sigma(n)}}\right) M_n\left(g_{\sigma(1)}, ..., g_{\sigma(n)}\right), \tag{7.17}$$

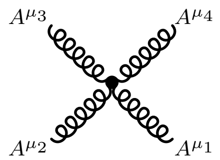
where the sum is over all non-cyclic permutations of the external gluon legs, i,e. $\sigma \in S_n/Z_n$. The partial amplitudes in these decompositions are computed from Feynman diagrams with a fixed ordering of the external gluons using the Feynman rules



$$: -\mathrm{i}\gamma^\mu\,, \tag{7.18}$$



$$:\mathrm{i}\Big[g^{\mu_1\mu_2}(k_1 - k_2)^{\mu_3} + g^{\mu_2\mu_3}(k_2 - k_3)^{\mu_1} + g^{\mu_3\mu_1}(k_3 - k_1)^{\mu_2}\Big] \tag{7.19}$$



$$:\mathrm{i}(2g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_2\mu_3}g^{\mu_1\mu_4}) \tag{7.20}$$

Some remarks on the derivation of the decompositions (7.16) and (7.17):

- The colour structures in the decompositions (7.16) and (7.17) arise by combining the colour-ordered three-and four gluon vertices because the Fierz identities (4.104) imply

$$\mathrm{tr}[AT^a]\,\mathrm{tr}[BT^a] = \frac{1}{2}\,\mathrm{tr}[AB] - \frac{1}{2N_c}\,\mathrm{tr}[A]\,\mathrm{tr}[B] \tag{7.21}$$

$$(AT^aB)^i{}_j\,\mathrm{tr}[CT^a] = \frac{1}{2}(ACB)^i{}_j - \frac{1}{2N_c}(AB)^i{}_j\,\mathrm{tr}[C] \tag{7.22}$$

- The colour suppressed term drops out in a contraction with a structure constant:

$$\mathrm{tr}[AT^c]f^{abc} \propto \mathrm{tr}[AT^c]\,\mathrm{tr}[[T^a, T^b]T^c] = \frac{1}{2}\,\mathrm{tr}[[T^a, T^b]A] - \frac{1}{2N_c}\,\mathrm{tr}[A]\,\mathrm{tr}[[T^a, T^b]] \tag{7.23}$$

  Therefore, the $1/N_c$ term drops out for purely gluonic amplitudes and amplitudes with one quark pair where every internal gluon line must connect to at least one purely gluonic vertex.

- A factor of 2 cancels whenever the colour structures of two vertices are combined with the identities (7.21). For purely gluonic amplitudes, the remaining factor of 2 is included by convention in the definition of the partial amplitudes.

For amplitudes with two quark pairs the color structure is more complicated since also contributions suppressed by the number of colors N have to be taken into account. For two different quark flavors Q and q the decomposition can be written as

$$
\begin{aligned}
\mathcal{A}_{n+4}(\bar{Q}_{\bar{p}}, Q_p, \bar{q}_{\bar{k}}, q_k, g_1, \dots g_n) = & \frac{g_s^{n-2}}{2} \sum_{i=0}^{n} \sum_{\sigma \in S_{1,i}} \sum_{\sigma \in S_{i+1,n}} \\
\times \Big[& (T^{a_{\sigma(2)}} ... T^{a_{\sigma(i)}})_{i_{\bar{p}} j_k} \; (T^{a_{\sigma(i+1)}} ... T^{a_{\sigma(n)}})_{j_{\bar{k}} i_p} \, A_n(\bar{Q}_{\bar{p}}, g_1, , \dots g_i, q_k, \bar{q}_{\bar{k}}, g_{i+1}, ..., g_n, Q_p) \\
- \frac{1}{N} & \; (T^{a_{\sigma(2)}} ... T^{a_{\sigma(i)}})_{i_{\bar{p}} i_p} \; (T^{a_{\sigma(i+1)}} ... T^{a_{\sigma(n)}})_{j_{\bar{k}} j_k} \, B_n(\bar{Q}_{\bar{p}}, g_1, , \dots g_i, Q_p; \bar{q}_{\bar{k}}, g_{i+1}, ..., g_n, q_k) \Big]
\end{aligned}
\tag{7.24}
$$

In the cases $i = 0$ and $i = n$ one of the strings of generators reduces to a Kronecker delta. For amplitudes with two pairs of identical quark flavors one has to subtract the right hand side after exchanging $Q_p \leftrightarrow q_k$.

Chapter 8

Born amplitudes

8.1 General considerations

Simplifying notation

- Consider amplitudes where all particles are outgoing with momenta k_i.
- Denote spinors for momentum k_i by $|i\pm\rangle$.

Relations among partial amplitudes

The number of helicity amplitudes that have to be computed can be reduced by using relations among different partial amplitudes. Some examples for such relations for the purely gluonic amplitudes are

- Cyclicity:
$$M_n\left(g_1, ..., g_n\right) = M_n\left(g_n, g_1, ..., g_{n-1}\right) \tag{8.1}$$
This follows from the cyclicity of the traces in the colour decomposition (7.17)
- Reflection identity
$$M_n\left(g_1, ..., g_n\right) = (-1)^n M_n\left(g_n, g_{n-1}, ..., g_1\right). \tag{8.2}$$
This follows from the fact that the colour-ordered three- and four-point vertices have the same properties.
- Parity:
$$M_n\left(g_1^{\lambda_1}, ..., g_n^{\lambda_n}\right) = (-1)^n M_n^*\left(g_1^{-\lambda_1}, ..., g_n^{-\lambda_n}\right) \tag{8.3}$$
The sign arises from the explicit factors of i in the colour ordered Feynman rules. (Recall that the Feynman rules compute iM, so one overall factor of i is removed). For real momenta, the complex conjugation is simply implemented using the identity for spinor products
$$\langle pk\rangle^* = [kp] \tag{8.4}$$

8.1.1 Choice of reference spinors

Calculations of scattering amplitudes with gluons can be simplified by choosing the arbitrary reference spinors $|q\pm\rangle$ in the polarization vectors

$$\epsilon_+^{\mu,*}(k,q) = \frac{\langle q|\,\gamma^\mu |k]}{\sqrt{2}\,\langle qk\rangle}\,, \qquad \epsilon_-^{\mu,*}(k,q) = \frac{[q|\gamma^\mu\,|k\rangle}{\sqrt{2}\,[kq]} \tag{8.5}$$

appropriately. Recall that gauge invariance implies that

- different reference spinors $|q_i\pm\rangle$ can be used for different gluons i
- the same choice of the $|q_i\pm\rangle$ has to be used in all diagrams contributing to a helicity amplitude
- different choices of the $|q_i\pm\rangle$ can be used for different helicity amplitudes.

For polarization vectors with different momenta and reference spinors one obtains the scalar products

$$\epsilon_+^*(k_1,q_1)\cdot\epsilon_+^*(k_2,q_2) = \frac{\langle q_1q_2\rangle\,[21]}{\langle q_11\rangle\,\langle q_22\rangle} \tag{8.6}$$

$$\epsilon_+^*(k_1,q_1)\cdot\epsilon_-^*(k_2,q_2) = \frac{\langle q_12\rangle\,[q_21]}{\langle q_11\rangle\,[2q_2]} \tag{8.7}$$

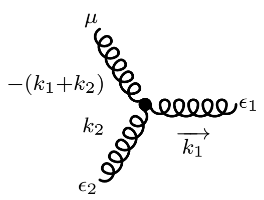
$$\epsilon_-^*(k_1,q_1)\cdot\epsilon_-^*(k_2,q_2) = \frac{[q_1q_2]\,\langle 21\rangle}{[1q_1]\,[2q_2]} \tag{8.8}$$

From these expressions one observes that many scalar products of polarization vectors can be eliminated by a clever choice of the reference spinors:

- Scalar products of polarization vectors with the same helicity vanish if the same reference spinors are used for gluons with the same helicity.
- Scalar products of polarization vectors with opposite helicity vanish if the momentum spinor of gluon 1 is used as the reference spinor of gluon 2 (or vice versa).

Three-gluon vertex

Consider the contraction of the gluon vertex with two external polarization vectors:



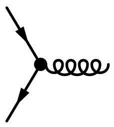
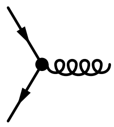
$$:\ \mathrm{i}\Big[(\epsilon_1\cdot\epsilon_2)(k_1-k_2)^\mu + 2\epsilon_2^\mu(k_2\cdot\epsilon_1) - 2\epsilon_1^\mu(k_1\cdot\epsilon_2)\Big] \tag{8.9}$$

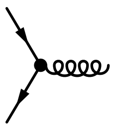
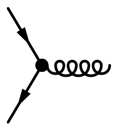
Using the above expressions for the scalar product of two polarization vectors and the fact that $q\cdot\epsilon(k,q)=0$ we see that

- for two neighboring gluons with opposite helicities, the three-gluon vertex vanishes if the momentum spinor of gluon 1 is used as the reference spinor of gluon 2 and vice versa.

Quark-gluon vertex

Contracting the quark-gluon vertex with one spinor of an outgoing antiquark and a gluon polarization vector gives the combinations:

$\bar{q}^-_{k_1}$  $g^+_{k_2}$: $\quad -\mathrm{i}\sqrt{2}|2]\dfrac{\langle q_2 1\rangle}{\langle q_2 2\rangle}$, $\qquad$ $\bar{q}^-_{k_1}$  $g^-_{k_2}$: $\quad -\mathrm{i}\sqrt{2}|q_2]\dfrac{\langle 21\rangle}{[2q]}$ (8.10)

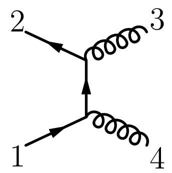
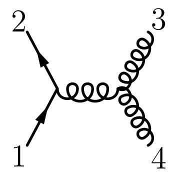
$\bar{q}^+_{k_1}$  $g^+_{k_2}$: $\quad -\mathrm{i}\sqrt{2}\,|q_2\rangle\,\dfrac{[21]}{\langle q_2 2\rangle}$, $\qquad$ $\bar{q}^+_{k_1}$  $g^-_{k_2}$: $\quad -\mathrm{i}\sqrt{2}\,|2\rangle\,\dfrac{[q_2 1]}{[2q]}$ (8.11)

From these results we observe

- For a neighbouring quark and gluon with opposite helicities, the quark gluon vertex vanishes if the gluon reference spinor is chosen as the momentum spinor of the quark.

8.2 2-quark 2-gluon amplitude

Two diagrams contribute to the colour-ordered $q\bar{q}gg$ amplitude:

$M(\bar{q}_1, q_2, g_3, g_4) =$  $+$  (8.12)

The only non-vanishing helicity amplitudes are those with one positive helicity quark (or antiquark) and gluon and one negative helicity quark (or antiquark) and gluon. This can be seen as follows:

- The quarks must have opposite helicities because of the coupling through a vector current. (The second diagram involves the spinor product $(\bar{u}_2\gamma^\mu v_1)$, the second one the spinor chain $(\bar{u}_2\gamma^\mu(\not{k}_1+k_4)\gamma^\nu v_1)$ which both vanish for spinors with equal helicity).

- For two gluons with equal positive (negative) helicity the amplitude vanishes according to the observations from Section 8.1.1 as can be seen by choosing both reference spinors as the momentum spinor of the negative (positive) helicity quark.

We will first compute the amplitude $M(\bar{q}_1^-, q_2^+, g_3^+, g_4^-)$. From the arguments of Section 8.1.1 we see that the s-channel gluon exchange diagram vanishes if we chose the reference spinors as $|q_3\rangle = |4\rangle$ and $|q_4] = |3]$. The polarization vectors then read:

$$\epsilon_+^{\mu,*}(k_3, q_3) = \frac{\langle 4|\,\gamma^\mu |3]}{\sqrt{2}\,\langle 43\rangle}\,, \qquad \epsilon_-^{\mu,*}(k_4, q_4) = \frac{[3|\gamma^\mu\,|4\rangle}{\sqrt{2}\,[43]} \tag{8.13}$$

For this choice of reference spinors, the amplitude is given just by the t-channel diagram:

$$\begin{aligned} \mathrm{i}M(\bar{q}_1^-, q_2^+, g_3^+, g_4^-) &= \mathrm{i}^3[2|\not\epsilon_3^+ \frac{\not k_2 + \not k_3}{(k_2+k_3)^2}\not\epsilon_4^-\,|1\rangle \\ &= \frac{2\mathrm{i}^3}{[23]\,\langle 32\rangle\,\langle 43\rangle\,[43]}\,[23]\,\langle 4|\,(\not k_2 + \not k_3)|3]\,\langle 41\rangle \\ &= -2\mathrm{i}\frac{\langle 42\rangle\,[23]\,\langle 41\rangle}{\langle 32\rangle\,\langle 43\rangle\,[43]} \\ &= 2\mathrm{i}\frac{\langle 14\rangle^2\,\langle 24\rangle}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle} \end{aligned} \tag{8.14}$$

In the last step we have used momentum conservation in the form

$$\langle 12\rangle\,[23] = -\,\langle 14\rangle\,[43]\,. \tag{8.15}$$

The amplitude with reversed helicities of the quarks is computed in the same way:

$$\begin{aligned} M(\bar{q}_1^+, q_2^-, g_3^+, g_4^-) &= \mathrm{i}^3\,\langle 2|\,\not\epsilon_3^+ \frac{(\not k_2 + \not k_3)}{(k_2+k_3)^2}\not\epsilon_4^-|1] \\ &= -\frac{2\mathrm{i}}{[23]\,\langle 32\rangle\,\langle 43\rangle\,[43]}\,\langle 24\rangle\,[3|(\not k_2 + k_3)\,|4\rangle\,[31] \\ &= -2\mathrm{i}\frac{\langle 24\rangle\,[32]\,\langle 24\rangle\,[31]}{[23]\,\langle 32\rangle\,\langle 43\rangle\,[43]} \\ &= -2\mathrm{i}\frac{\langle 24\rangle^3}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle} \end{aligned} \tag{8.16}$$

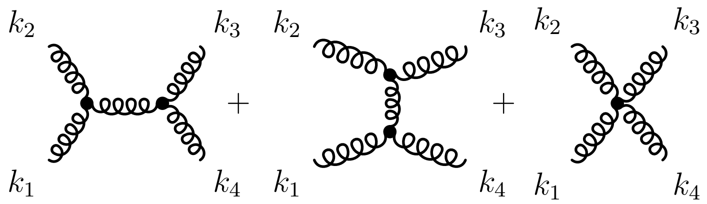
where we have used also

$$[31]\,\langle 12\rangle = -\,[34]\,\langle 42\rangle \tag{8.17}$$

8.3 4 gluon amplitude

Three diagrams contribute to the colour-ordered 4-gluon amplitude:

$$\mathrm{i}M(g_{k_1}, g_{k_2}, g_{k_3}, g_{k_4}) = \tag{8.18}$$



The number of independent helicity amplitudes to be calculated is reduced by some considerations:

- The amplitude where all gluons have the same helicity vanish. This can be seen by choosing the same reference spinor for all gluons so that all scalar products of polarization vectors vanish.

- The amplitude where only one gluon has a different helicity than the others vanish as well. This can be seen by choosing the reference spinors of all the equal-helicity gluons in terms of the momentum spinor of the remaining gluon, so all the scalar products of polarization vectors vanish.

- Therefore one only needs to calculate amplitudes with two positive and two negative helicities. Using the properties of the partial amplitudes discussed in Section 8.1 one only needs to calculate two helicity amplitudes, e.g. $M(g^-, g^-, g^+, g^+)$ and $M(g^-, g^+, g^-, g^+)$.

We begin with the amplitude $M(g^-_{k_1}, g^-_{k_2}, g^+_{k_3}, g^+_{k_4})$. The calculation is simplified by a suitable choice of the reference spinors:

- Following the discussion in Section 8.1.1 we use one reference spinor $|q]$ for gluons 1 and 2 and one reference spinor $|q\rangle$ for gluons 3 and 4. Then the scalar products of polarization vectors with the same helicity vanish:

$$\epsilon_1^{-*} \cdot \epsilon_2^{-*} = \epsilon_3^{+*} \cdot \epsilon_4^{+*} = 0 \tag{8.19}$$

- Choosing $|q\rangle = |1\rangle$ and $|q] = |4]$ also

$$\epsilon_1^{-*} \cdot \epsilon_{3/4}^{+*} = 0 \qquad\qquad \epsilon_4^{+*} \cdot \epsilon_{1/2}^{-*} = 0 \tag{8.20}$$

  As discussed in 8.1.1, for this choice the three-gluon vertex involving gluons 1 and 4 vanishes.

- The only non-vanishing scalar product of polarization vectors is given by

$$(\epsilon_2^{-*} \cdot \epsilon_3^{+*}) = \frac{\langle 12\rangle\, [43]}{\langle 13\rangle\, [24]} \tag{8.21}$$

- The diagram with the four-point vertex vanishes since it involves the scalar products

$$2(\epsilon_1^{-*} \cdot \epsilon_3^{+*})(\epsilon_2^{-*} \cdot \epsilon_4^{+*}) - (\epsilon_1^{-*} \cdot \epsilon_2^{-*})(\epsilon_3^{+*} \cdot \epsilon_4^{+*}) - (\epsilon_2^{-*} \cdot \epsilon_3^{+*})(\epsilon_1^{-*} \cdot \epsilon_4^{+*}) \tag{8.22}$$

  where all the terms vanish.

For the above choice of reference spinors only the first diagram contributes to the amplitude and one obtains

$$\begin{aligned}
\mathrm{i}M(g^-_{k_1}, g^-_{k_2}, g^+_{k_3}, g^+_{k_4}) =&(\mathrm{i})^2\left[2\epsilon_2^{-*,\mu}(k_2\cdot\epsilon_1^{-*}) - 2\epsilon_1^{-*,\mu}(k_1\cdot\epsilon_2^{-*})\right]\\
&\times\frac{(-\mathrm{i})}{(k_1+k_2)^2}\left[2\epsilon^{+*}_{4,\mu}(k_4\cdot\epsilon_3^{+*}) - 2\epsilon^{+*}_{3,\mu}(k_3\cdot\epsilon_4^{+*})\right]\\
&= -4\mathrm{i}\frac{(k_2\cdot\epsilon_1^{-*})(k_3\cdot\epsilon_4^{+*})(\epsilon_2^{-*}\cdot\epsilon_3^{+*})}{2(k_1\cdot k_2)}\\
&= -2\frac{\mathrm{i}}{\langle 12\rangle\,[21]}\frac{k_2^\mu[4|\gamma_\mu\,|1\rangle}{[14]}\frac{k_3^\nu\,\langle 1|\,\gamma_\nu|4]}{\langle 14\rangle}\frac{\langle 12\rangle\,[43]}{\langle 13\rangle\,[24]}\\
&= -2\frac{\mathrm{i}}{\langle 12\rangle\,[21]}\frac{[42]\,\langle 21\rangle}{[14]}\frac{\langle 13\rangle\,[34]}{\langle 14\rangle}\frac{\langle 12\rangle\,[43]}{\langle 13\rangle\,[24]}\\
&= 2\mathrm{i}\frac{[43]^2}{[21]\,[14]}\frac{\langle 12\rangle}{\langle 14\rangle}\\
&= 2\mathrm{i}\frac{\langle 12\rangle^3}{\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}
\end{aligned}\tag{8.23}$$

Here momentum conservation was used in the forms

$$[43]\,\langle 32\rangle = -\,[41]\,\langle 12\rangle \tag{8.24}$$

and $(k_3+k_4)^2 = (k_1+k_2)^2$ so that

$$[34]\,\langle 43\rangle = [12]\,\langle 21\rangle \tag{8.25}$$

As a second independent helicity amplitude we can take $M(g^+_{k_1}, g^-_{k_2}, g^+_{k_3}, g^-_{k_4})$. Choosing the reference spinors as

$$|q_1\rangle = |q_3\rangle = |4\rangle \qquad\qquad |q_2] = |q_4] = |1] \tag{8.26}$$

again all scalar products of polarization vectors vanish apart from

$$(\epsilon_2^{-*}\cdot\epsilon_3^{+*}) = \frac{\langle 42\rangle\,[13]}{\langle 43\rangle\,[21]}. \tag{8.27}$$

Furthermore the three-gluon vertex involving gluons 1 and 4 is again zero so only the s-channel diagram contributes. We get

$$
\begin{aligned}
\mathrm{i}M(g^+_{k_1}, g^-_{k_2}, g^+_{k_3}, g^-_{k_4}) &= -4\mathrm{i}\frac{(k_2\cdot\epsilon_1^{+*})(k_3\cdot\epsilon_4^{-*})(\epsilon_2^{-*}\cdot\epsilon_3^{+*})}{2(k_1\cdot k_2)} \\
&= -2\frac{\mathrm{i}}{\langle 12\rangle\,[21]}\frac{k_2^\mu\,\langle 4|\,\gamma_\mu|1]}{\langle 41\rangle}\frac{k_3^\nu[1|\gamma_\nu\,|4\rangle}{[41]}\frac{\langle 42\rangle\,[13]}{\langle 43\rangle\,[21]} \\
&= -2\frac{\mathrm{i}}{\langle 12\rangle\,[21]}\frac{\langle 42\rangle\,[21]}{\langle 41\rangle}\frac{[13]\,\langle 34\rangle}{[41]}\frac{\langle 42\rangle\,[13]}{\langle 43\rangle\,[21]} \\
&= 2\mathrm{i}\frac{[13]^2}{[12]\,[14]}\frac{\langle 24\rangle^2}{\langle 12\rangle\,\langle 41\rangle} \\
&= 2\mathrm{i}\frac{\langle 24\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}
\end{aligned}
\tag{8.28}
$$

8.4 Properties of Multi-leg amplitudes

8.4.1 Maximally-helicity violating amplitudes

In 1986 Parke and Taylor conjectured a very simple form for some helicity amplitudes with an arbitrary numbers of gluons [15], which is a simple generalization of the four-point results obtained so far. Amplitudes where all gluons or all but one gluon have the same helicity vanish ($\Rightarrow$ homework):

$$
M_n(g_1^+,\ldots,\ldots g_n^+) = 0 \qquad\qquad M_n(g_1^+,\ldots,g_j^-,\ldots g_n^+) = 0 \tag{8.29}
$$

The gluonic amplitudes where all but two gluons have the same helicity are called "maximally helicity violating" (MHV) amplitudes. For those, Parke and Taylor conjectured the formula:

$$
M_n(g_1^+,\ldots,g_i^-,\ldots,g_j^-,\ldots g_n^+) = 2^{n/2-1}\frac{\langle ij\rangle^4}{\langle 12\rangle\,\langle 23\rangle\ldots\langle n1\rangle} \tag{8.30}
$$

where all the gluons not shown explicitly have positive helicity. The corresponding amplitudes with two positive helicities and an arbitrary number of negative helicity gluons is obtained by complex conjugation.

The MHV amplitudes with a quark-antiquark pair and an arbitrary number of gluons are related in a simple way to the all-gluon amplitudes:

$$
M_n(\bar{q}_1^-, q_2, g_3^+,\ldots g_j^-,\ldots,g_n^+) = -\frac{\langle 2j\rangle}{\langle 1j\rangle}M_n(g_1^-, g_2^+,\ldots g_j^-,\ldots,g_n^+) \tag{8.31}
$$

$$
= -2^{n/2-1}\frac{\langle 1j\rangle^3\,\langle 2j\rangle}{\langle 12\rangle\,\langle 23\rangle\ldots\langle n1\rangle} \tag{8.32}
$$

This relation can be derived e.g. by embedding Yang-Mills theory in a supersymmetric theory, see [17] for a review.

The MHV amplitudes cover all helicity amplitudes contributing to four- and five-point amplitudes. Especially for the five-point case this result is a drastic simplification compared to the computation using textbook methods. Starting from six points, so-called "next-to MHV" amplitudes with three opposite-helicity gluons arise. The use of the simple form of the MHV amplitudes for the computation of more complicated amplitudes has been limited until a new recursive construction of amplitudes was found in 2004, which uses on-shell amplitudes as input. This is discussed in Chapter 9.

8.4.2 Little-group scaling

Recall that the momentum spinors associated to a light-like momentum p^μ are only determined up to a scaling

$$p^A \to z\, p^A\,, \qquad\qquad p^{\dot{A}} \to z^{-1}\, p^{\dot{A}} \tag{8.33}$$

Under this scaling the polarization spinors and vectors of outgoing quarks and gluons behave as

$$|k+\rangle \to z^{-1}\, |k+\rangle\ , \qquad\qquad |k-\rangle \to z\, |k-\rangle\ , \tag{8.34}$$

$$\epsilon_+^{*\mu} \to z^{-2}\epsilon_+^{*\mu}\,, \qquad\qquad \epsilon_-^{*\mu} \to z^{2}\epsilon_-^{*\mu} \tag{8.35}$$

Since a helicity amplitude is a function of the momenta of the external particles, which are invariant under the scaling of the spinors, and the polarization vectors we have

$$M_n(\phi_1^{\lambda_1}, \phi_2^{\lambda_2}, \dots \phi_n^{\lambda_n}) \to z^{-2(\lambda_1+\dots\lambda_n)} M_n(\phi_1^{\lambda_1}, \phi_2^{\lambda_2}, \dots \phi_n^{\lambda_n}) \tag{8.36}$$

One sees that the MHV amplitudes satisfy this identity.

The identity can also be written in terms of the "helicity operators"

$$h_i = \frac{1}{2}\left(k_i^A \frac{\partial}{\partial k_i^A} - k_i^{\dot{A}} \frac{\partial}{\partial k_i^{\dot{A}}} \right). \tag{8.37}$$

Since the source of the helicity dependence comes from the external wave-functions alone, we have the identities

$$h_i M_n(\phi_1^{\lambda_1}, \phi_2^{\lambda_2}, \dots \phi_n^{\lambda_n}) = -\lambda_i M_n(\phi_1^{\lambda_1}, \phi_2^{\lambda_2}, \dots \phi_n^{\lambda_n}) \tag{8.38}$$

for each leg individually.

8.4.3 Soft and collinear limits

Soft limit

For the limit where one of the momenta k_l of the positive helicity gluons in an MHV amplitude becomes very small, $k_l \to \lambda k_l$, $\lambda \to 0$ we have

$$M_n(g_1^+, \dots g_i^-, \dots g_l^+, \dots g_j^- \dots) \to \lambda^{-2} S^{(0)+}_{l-1,l,l+1} M_n(g_1^+, \dots g_i^-, \dots g_{l-1}^+, g_{l+1}^+, \dots g_j^{-1} \dots) \tag{8.39}$$

with

$$S^{(0)+}_{ijk} = \frac{\langle ik\rangle}{\langle ij\rangle\,\langle jk\rangle} \tag{8.40}$$

This relation is actually true for all tree amplitudes, not just MHV amplitudes as can be shown either from a Feynman diagram argument or using the recursive constructions discussed in subsequent chapters. For a soft negative-helicity gluon the MHV amplitudes vanish. In general the relation is

$$M_n(g_1,\ldots\ldots g_l^-,\ldots) \to \lambda^{-2} S^{(0)-}_{l-1,l,l+1} M_n(g_1^+,\ldots g_i^-,\ldots g_{l-1}^+, g_{l+1}^+,\ldots g_j^{-1}\ldots) \tag{8.41}$$

with

$$S^{(0)-}_{ijk} = \frac{[ki]}{[kj]\,[ji]} \tag{8.42}$$

Collinear limit

Two particles i and j are said to become collinear (denoted as $i \parallel j$) if their momenta become proportional, $k_i \propto k_j$. The momenta of the two collinear particles can be parameterized as

$$k_i^\mu = zP^\mu + k_\perp^\mu - \frac{k_\perp^2}{2\,z\,(P\cdot n)}n^\mu \qquad k_j^\mu = (1-z)P^\mu - k_\perp^\mu - \frac{k_\perp^2}{2\,(1-z)\,(P\cdot n)}n^\mu \tag{8.43}$$

with $P^2 = n^2 = 0$, $(n\cdot P) \neq 0$ and $k_\perp \cdot n = k_\perp \cdot P = 0$. We have

$$(k_i + k_j)^2 = 2(k_i \cdot k_j) = -\frac{k_\perp^2}{z(1-z)} \tag{8.44}$$

In the collinear limit $k_\perp^2 \to 0$ one gets $k_i + k_j \to P$ with $P^2 = 0$.

In the helicity method, the collinear limit can be formulated as the limit

$$|k_i\pm\rangle \to \sqrt{z}\,|P\pm\rangle \qquad\qquad |k_j\pm\rangle \to \sqrt{(1-z)}\,|P\pm\rangle \tag{8.45}$$

Note that

$$\langle k_i k_j\rangle \sim [k_j k_i] \sim \sqrt{-\frac{k_\perp^2}{z(1-z)}} \tag{8.46}$$

so the limit $k_\perp \to 0$ can not be taken in the spinor products involving both collinear momenta.

From the MHV amplitudes we observe

$$M_n(g_1^+, g_2^+\ldots, g_i^-,\ldots, g_j^-,\ldots g_n^+) \xrightarrow{1\|2} \frac{1}{\sqrt{z(1-z)}\,\langle 12\rangle} M_{n-1}(g_P^+\ldots, g_i^-,\ldots, g_j^-,\ldots g_n^+) \tag{8.47}$$

$$M_n(g_1^+,\ldots, g_i^-,\ldots, g_j^-,\ldots g_n^+) \xrightarrow{(i-1)\|i} = \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle (i-1)i\rangle} M_{n-1}(g_1^+\ldots, g_P^-,\ldots, g_j^-,\ldots g_n^+) \tag{8.48}$$

$$M_n(g_1^+, \ldots, g_i^-, \ldots, g_j^-, \ldots g_n^+) \xrightarrow{i\|(i+1)} = \frac{z^2}{\sqrt{z(1-z)}\,\langle i(i+1)\rangle} M_{n-1}(g_1^+ \ldots, g_P^-, \ldots, g_j^-, \ldots g_n^+) \tag{8.49}$$

$$M_n(g_1^+, \ldots, g_i^-, g_{i+1}^-, \ldots g_n^+) \xrightarrow{i\|(i+1)} 0 \tag{8.50}$$

It can be shown that the collinear limit of general tree amplitudes is given by the factorization

$$M_n(\ldots, g_i^{\lambda_i}, g_{i+1}^{\lambda_{i+1}} \ldots) \xrightarrow{i\|(i+1)} \sum_{\lambda} \mathrm{Split}_{-\lambda}(z, g_i^{\lambda_i}, g_{i+1}^{\lambda_{i+1}}) M_{n-1}(\ldots, g_P^{\lambda}, \ldots) \tag{8.51}$$

The following results for the **splitting functions** read off from the MHV amplitudes turn out to be correct for all Born amplitudes

$$\mathrm{Split}_-(z, g_i^+, g_{i+1}^+) = \frac{1}{\sqrt{z(1-z)}\,\langle i(i+1)\rangle} \tag{8.52}$$

$$\mathrm{Split}_+(z, g_i^+, g_{i+1}^+) = 0 \tag{8.53}$$

$$\mathrm{Split}_+(z, g_i^+, g_{i+1}^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle i(i+1)\rangle} \tag{8.54}$$

$$\mathrm{Split}_+(z, g_i^-, g_{i+1}^+) = \frac{z^2}{\sqrt{z(1-z)}\,\langle i(i+1)\rangle} \tag{8.55}$$

The splitting amplitudes for opposite helicities can be obtained by complex conjugation. In contrast to the soft limit, the splitting functions depend on the spin of the collinear particles. The corresponding splitting functions for amplitudes involving quarks can be obtained analogously from the MHV amplitudes.

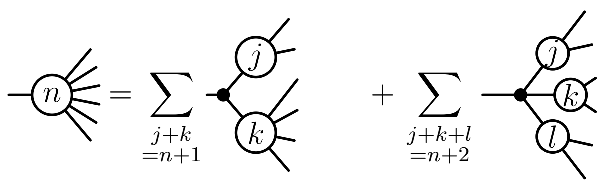
8.5 Berends-Giele recursion relations

The Parke-Taylor formula for the MHV amplitudes (8.30) was proven by Berends and Giele [16] using a recursive construction. This involves matrix elements with one off-shell leg where the polarization vector or spinor of the corresponding particle is stripped off. The relation of these off-shell matrix elements to the scattering amplitudes is

$$M_n(g_1, ..., g_{n-1}, g_n) = \epsilon_\mu^*(k_n, q) M_n^\mu(g_1, ..., g_{n-1}, \widehat{g}_{k_n})|_{k_n^2=0} \tag{8.56}$$

where the hat denotes the off-shell particle.

The one-particle off-shell matrix elements satisfy a recursion relation which can be represented graphically as



The corresponding equation reads

$$\begin{aligned}M_n^\mu(g_1, ..., g_{n-1}, \widehat{g}_{k_n}) = \sum_{j=2}^{n-1} V_3^{\mu\nu\rho}(-k_{1,j}, -k_{j,n}, k_n)\, \frac{-\mathrm{i}}{k_{1,j}^2} M_{j,\nu}(g_1, ..., g_{j-1}, \widehat{g}_{-k_{1,j}}) \\ \frac{-\mathrm{i}}{k_{j,n}^2} M_{n-j+1,\rho}(g_j, ..., g_{n-1}, \widehat{g}_{-k_{j,n}}) \\ + \sum_{j=2}^{n-2}\sum_{k=j}^{n-1} V_4^{\mu\nu\rho\sigma} \frac{-\mathrm{i}}{k_{1,j}^2} M_{j,\nu}(g_1, ..., g_{j-1}, \widehat{g}_{-k_{1,j}}) \\ \frac{-\mathrm{i}}{k_{j,k}^2} M_{k-j+1,\rho}(g_j, ..., g_{k-1}.\widehat{g}_{-k_{j,k}}) \frac{-\mathrm{i}}{k_{j,n}^2} M_{n-k+1,\rho}(g_k, ..., g_{n-1}, \widehat{g}_{-k_{k,n}})\end{aligned} \tag{8.57}$$

Here we have defined

$$k_{i,j} = \sum_{l=i}^{j-1} k_l \tag{8.58}$$

and the vertex functions are those of the colour-ordered Feynman rules given in Section 7.2.2. The recursion starts with the two-point amplitudes, which are given by the gluon polarization vector multiplied by the inverse propagator:

$$M_2(g_i, \widehat{g}_{-i}) = \mathrm{i}k_i^2 \epsilon(k_i, q)^\mu. \tag{8.59}$$

For the MHV amplitudes, the structure of the recursion relation simplifies since one can show that the quartic vertex does not contribute. By induction it can be shown that the off-shell matrix element with only positive-helicity gluons is given by

$$M_n\left(g_1^+, ..., g_{n-1}^+, \widehat{g_n^-}\right) = 2^{n/2-2}(-ik_{1,n}^2) \frac{\langle q|\gamma^\mu \not{k}_{1,n}|q\rangle}{\langle q1\rangle \langle 12\rangle \ldots \langle (n-2)(n-1)\rangle \langle (n-1)q\rangle}. \tag{8.60}$$

For the corresponding off-shell matrix element with one negative-helicity gluon one can also obtain a closed expression that can be used to proof the formula for the MHV amplitude [17].

Chapter 9

On-shell recursion relations

A few years ago, Britto,Cachazo and Feng (BCF) [12] discovered a method to construct on-shell tree-amplitudes in a recursive fashion using only on-shell amplitudes with fewer legs as input. Originally these relations were found from the cancellation of infrared singularities in one-loop amplitudes in maximally supersymmetric Yang-Mills theory, but later proven by Britto,Cachazo, Feng and Witten (BCFW) in [13] using only complex analysis and factorization properties of scattering amplitudes. This allowed to extend the method also to other theories including theories with massive particles and also gravity. The on-shell recursion relations are by now discussed in textbooks [1] and are reviewed in several lecture notes and reviews including [8, 10, 9, 11].

9.1 BCFW recursion relation

9.1.1 Statement of the on-shell recursion relations

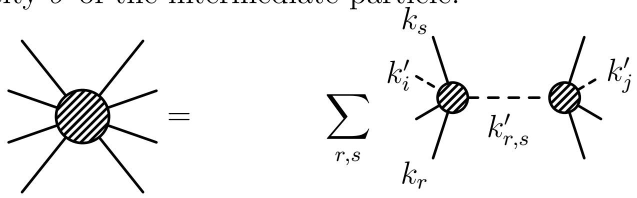
In order to formulate the on-shell recursion relations, we consider a tree-level scattering amplitude in QCD with n external particles and pick out two momenta of the external particles, k_i and k_j. The recursion relation then expresses the n-particle on-shell tree-amplitude in terms of of two on-shell sub-amplitudes with fewer external particles, with the special momenta k_i and k_j deformed in a particular way to be explained below (denoted by a prime):

$$\mathrm{i}M_n(1,2,\ldots,n) = \sum_{P(i,j),\sigma} \mathrm{i}M(r,\ldots i',\ldots,s,-k_{r,s}'^{\sigma})\frac{\mathrm{i}}{k_{r,s}^2}\mathrm{i}M(k_{r,s}'^{-\sigma},s{+}1,\ldots,j',\ldots,r{-}1) \quad (9.1)$$

where

$$k_{i,j} = \sum_{l=1}^{j} k_l. \quad (9.2)$$

The sum in (9.1) is over all partitions the external particles into two sets $P(i,j) = \{(r,\ldots,(s-1)),(s,\ldots r-1)\}$ so that momentum k_i is in the first set and k_j in the second one, and over the helicity σ of the intermediate particle.



For colour-ordered amplitudes, also only ordered partitions contribute. In the subamplitudes on the right hand side the two external momenta k_i and k_j are shifted by a light-like vector η:

$$k'_i = k_i + z\eta \quad , \quad k'_j = k_j - z\eta \tag{9.3}$$

that is orthogonal to both special momenta, $(\eta \cdot k_i) = (\eta \cdot k_j) = 0$. The vector η must be complex since these conditions cannot be satisfied three real four-vectors, apart from special kinematical points (for lightlike $k_{i/j}$). In this way, also the shifted momenta are on-shell, $k'^2_{i/j} = k^2_{i/j} = 0$ Note that momentum conservation is still satisfied by the shifted momenta, $k'_i + k'_j = k_i + k_j$. In each of the terms in the sum in (9.1) the constant z takes a different value, determined such that the internal momenta $k'_{r,s} = k_{r,s} + z\eta$ are on-shell:

$$(k_{r,s} + z_{rs}\,\eta)^2 = 0 \;\Rightarrow z_{rs} = -\frac{k^2_{r,s}}{2k_{r,s} \cdot \eta} \tag{9.4}$$

Then, as promised, all external momenta of the subamplitudes on the right-hand side of (9.1) are on-shell, albeit with some of the momenta turned complex. The choice of the special momenta $k_{i/j}$ has to satisfy to certain restrictions on the helicities of the particles i and j, as will become clear in the proof presented below, but are otherwise arbitrary. Choosing different momenta for the shifts therefore can result in different, equivalent representations of an amplitude.

The on-shell recursion relation can be viewed as a way to construct a tree-level amplitude entirely from it's multi-particle poles. In general, at a multi-particle pole $k^2_{r,s} - M^2 = 0$, a scattering amplitude in an arbitrary QFT factorizes into two subamplitudes [5, 1]

$$\mathrm{i}M(k_1, \ldots k_n) \xrightarrow{k^2_{r,s} \to M^2} \sum_{\lambda} \mathrm{i}M(k_r, \ldots k_s, -k^{\lambda}_{r,s}) \frac{\mathrm{i}}{k^2_{r,s} - M^2} \mathrm{i}M(k^{-\lambda}_{r,s}, k_{s+1}, \ldots, k_{r-1}) \tag{9.5}$$

This expression is reminiscent of the terms in the on-shell recursion, however here the momenta are not shifted into the complex plane. Nevertheless, the factorization (9.5) is an important ingredient in the proof of the recursion relation given below.

9.1.2 Implementing the shift

Most applications of the on-shell recursion relations have been within the spinor helicity formalism. In this notation, a general solution to the shift vector for lightlike k_i and k_j is given by

$$\eta^\mu = \frac{1}{2} [i|\gamma^\mu\,|j\rangle \tag{9.6}$$

Therefore the shift can be described entirely in terms of spinors:

$$\begin{aligned} |i'\rangle &= |i\rangle + z\,|j\rangle\,, \quad & |i'] &= |i] \\ |j'\rangle &= |j\rangle\,, & |j'] &= |j] - z|i] \end{aligned} \tag{9.7}$$

since

$$\begin{aligned} k_i'^{\mu} &= \frac{1}{2}[i|\gamma^\mu\,|i'\rangle = k_i^\mu + z\eta \\ k_j'^{\mu} &= \frac{1}{2}[j'|\gamma^\mu\,|j\rangle = k_i^\mu - z\eta \end{aligned} \tag{9.8}$$

If the shifted particles are gluons, their polarization vectors can simply be defined using the shifted spinors (9.7):

$$\begin{aligned} \epsilon_\mu^{+*}(k_i') &= \frac{\langle q|\,\gamma_\mu|i]}{\sqrt{2}\,\langle qi'\rangle}, \quad & \varepsilon_\mu^{-*}(k_i') &= \frac{[q|\gamma_\mu\,|i'\rangle}{\sqrt{2}\,[iq]}, \\ \epsilon_\mu^{+*}(k_j') &= \frac{\langle q|\,\gamma_\mu|j']}{\sqrt{2}\,\langle qi\rangle}, & \varepsilon_\mu^{-*}(k_j') &= \frac{[q|\gamma_\mu\,|j\rangle}{\sqrt{2}\,[j'q]}. \end{aligned} \tag{9.9}$$

For massless quarks, the shift of external spinors is given directly by (9.7).

9.1.3 Proof of the recursion relation

In the proof of the on-shell recursion relation given by Britto,Cachazo, Feng and Witten (BCFW) in [13] one defines a continuation $M(z)$ of scattering amplitudes by performing the shift (9.7) for aribtrary complex values of z. The physical amplitude is then given by the value $M(0)$. The BCFW argument can be used to show that the on-shell recursion relations hold for tree-amplitudes in any QFT, provided the amplitudes vanish for shifts with large z:

$$\lim_{z\to\infty} M_n(z) = 0 \tag{9.10}$$

As discussed below this condition is satisfied in gauge theories with matter with some restriction on the helicity of the shifted legs. It turns out that the condition (9.10) is even satisfied in gravity. The large z behaviour of gauge and gravity theories in any space-time dimensions is investigated in [18].

The next step in the proof is to note that (due to the properties of Feynman rules) the analytically continued tree-amplitudes $M(z)$ are analytic functions of z that only have **simple poles** in z. This is since the only poles of an amplitude[1] can arise from propagators. Since the shift vector is lightlike, the propagator of an internal massless line with momentum $K'_\alpha = K_\alpha + z\eta$ has a single pole at

$$K_\alpha'^2 = 0 \;\Rightarrow z_\alpha = -\frac{K_\alpha^2}{2\,K_\alpha\cdot\eta} \tag{9.11}$$

[1]Assuming the external states do not give rise to additional poles in z. The shifted gluon polarization vectors (9.9) do have poles that, however, are gauge dependent and therefore drop out of the amplitude.

If the relation (9.10) is satisfied, the function $M(z)/z$ vanishes when integrated over a circle with $|z| \to \infty$. On the other hand, this integral is given by the sum of the residues at the poles z_α and the residue at $z = 0$, which gives the physical amplitude:

$$0 = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z \frac{M_n(z)}{z} = M_n(0) + \sum_{\text{poles } z_\alpha} \mathrm{Res}_{z_\alpha} \frac{M_n(z)}{z} \tag{9.12}$$

The last step to derive the recursion formula is now to compute the residue of the amplitude at the poles z_α. Using the factorization (9.5), the amplitude at the pole z_{rs} factorizes according to

$$\begin{aligned}\lim_{z\to z_{rs}} \mathrm{i}M_n(z) &= \sum_\lambda \mathrm{i}M(k_r, \dots k'_i, \dots k'^{\lambda}_{r,s}) \frac{\mathrm{i}}{k^2_{r,s} + 2zk_{r,s}\cdot\eta} \mathrm{i}M(-k'^{-\lambda}_{r,s}, k_{s+1}, \dots k'_j, \dots) \\ &= -\frac{z_{rs}}{z - z_{rs}} \sum_\lambda \mathrm{i}M(k_r, \dots k'_i, \dots k'^{\lambda}_{r,s}) \frac{\mathrm{i}}{k^2_{r,s}} \mathrm{i}M(-k'^{-\lambda}_{r,s}, k_{s+1}, \dots k'_j, \dots)\end{aligned}$$

since

$$\frac{z_{rs}}{z - z_{rs}} = \frac{1}{z/z_{rs} - 1} = -\frac{k^2_{r,s}}{2z\,(k_{r,s}\cdot\eta) + k^2_{r,s}} \tag{9.13}$$

Poles in z can only occur if particles i and j are on different sides of the propagator since the shift vector drops out by momentum conservation if both shifted legs are on the same side. By convention, particle i has been assigned to the "left" subamplitude.

Inserting the factorized amplitude into the complex integral gives the physical amplitude $M(0)$ expressed as a sum over the poles

$$\begin{aligned}M_n(0) &= -\sum_{\text{poles } z_\alpha} \mathrm{Res}_{z_\alpha} \frac{M_n(z)}{z} = -\sum_{\text{Poles}} \lim_{z\to z_{rs}} \frac{z - z_{rs}}{z} M_n(z) \\ &= \sum_{\text{Poles}} \sum_\lambda M(k_r, \dots k'_i, \dots k'^{\lambda}_{r,s}) \frac{\mathrm{i}}{k^2_{r,s}} M(-k'^{-\lambda}_{r,s}, k_{s+1}, \dots k'_j, \dots)\end{aligned}$$

which is the on-shell recursion relation. It should be clear, that the recursion is valid in all theories where scattering amplitudes vanish for $z \to \infty$. Proving an on-shell recursion for a given theory is therefore reduced to verifying (9.10). The same argument given here also goes through for massive particles.

Behaviour of amplitudes for $z \to \infty$

We here give results on the large-z behaviour of scattering amplitudes in various theories. The detailed investigation of this topic is beyond the scope of this introduction. For the case of Yang-Mills theory we will discuss a simple scaling argument that was used in [13] to proof the validity of the recursion relation for the case where particle i is a gluon with positive helicity and particle j is a gluon with negative helicity. For other helicity combinations more sophisticated arguments are required. A general analysis for Yang-Mills and Gravity theories based on the background field method was then given in [18]

ϕ^4 theory The four-point function in ϕ^4 theory is obviously is given by a single diagram without any propagator and therefore goes to a constant for $z \to \infty$. In higher-point amplitudes there will always be diagrams where particles i and j meet at the same vertex so that these diagrams have no z dependence and we have in general

$$M_{\phi^4}(z) \to \text{const}, \tag{9.14}$$

so that on-shell recursion is not possible and ϕ^4 theory is not constructable from two-line shifts.

Yang-Mills theory We discuss a simple power-counting argument for Yang-Mills theory and then quote results from the literature that allow to improve the bounds further.

- The sources for z-dependence of the amplitude are: momentum-dependent triple gluon vertices, external polarization vectors (9.9) and z-dependent propagators.
- The contributions with the maximal positive power of z come from Feynman diagrams where gluons i and j are connected by a string of cubic vertices, which each potentially contribute a factor z while each propagator (in Feynman gauge) contributes a factor z^{-1}.

Diagrams with n gluon propagators separating legs i and j therefore scale like

$$M(z) \sim \underbrace{n\,\text{propagators}}_{z^{-n}} \times \underbrace{(n+1)\,\text{vertices}}_{z^{n+1}} \times \epsilon_i \times \epsilon_j \sim z \times \epsilon_i \times \epsilon_j \tag{9.15}$$

From the shifted polarization vectors (9.9) one finds the asymptotic behaviour under a shift with $z \to \infty$:

$$\begin{aligned} \epsilon^+_\mu(k'_i) &\sim \frac{1}{z}, \quad \varepsilon^-_\mu(k'_i) \sim z, \\ \epsilon^+_\mu(k'_j) &\sim z, \quad \varepsilon^-_\mu(k'_j) \sim \frac{1}{z}. \end{aligned} \tag{9.16}$$

For the various helicity combinations of the shifted gluons we therefore obtain the following estimates:

$$(i^+, j^-): \qquad M(z) \sim \frac{1}{z} \tag{9.17}$$
$$(i^\pm, j^\pm): \qquad M(z) \sim z \tag{9.18}$$
$$(i^-, j^+): \qquad M(z) \sim z^3 \tag{9.19}$$

The power-counting argument shows that the (i^+, j^-) shift leads to a valid recursion relation. More detailed analyses allow to show that the power-counting overestimates the true behaviour for the equal-helicity case and the amplitudes fall off as z^{-1} as well in this case. The results can be summarized as

- allowed shifts: (i^+, j^-), (i^+, j^+), (i^-, j^-)
- forbidden shifts: (i^-, j^+).

Quark amplitudes Adding quarks to pure Yang-Mills leads to some modifications:

- For massless quarks the external spinors are given by (9.7) so the behaviour at large z is
$$\begin{aligned} v^+(k_i') = |i] \sim 1, \qquad & v^-(k_i') = |i'] \sim z \\ v^+(k_j') = |j'] \sim z, \quad & v^-(k_j') = |j] \sim 1 \end{aligned} \tag{9.20}$$
- z-dependent quark propagators scale like z^0
- the quark-gluon vertex scales like z^0

This leads to the following modifications:

- Replacing a pair of external non-shifted lines along the flow of z through the diagram in (9.15) by external quarks replaces (say) m gluon propagators by quark propagators and $m{+}1$ triple gluon vertices by quark-gluon vertices, therefore improving the scaling estimate by a factor of z^{-1}.
- If one of the shifted legs is replaced by a quark, the behaviour of the polarization spinors is worse that that for gluons by a factor z, however this is canceled by an improvement z^{-1} from the vertex. Therefore also the shifts (g_i^+, q_j^-) and (q_i^+, g_j^-) are allowed.
- If two quark lines are shifted, the shift (q_i^+, q_j^-) is allowed, unless both quarks belong to the same fermion line since in this case the diagram scales like z^0.
- A more detailed analysis allows to show that also the combinations (g_i^+, g_j^+), (g_i^+, q_j^+) and (q_i^+, q_j^+) are allowed [19].

9.2 Applications

9.2.1 Building blocks: three-point amplitudes

Since the on-shell recursion allows to construct four-point functions from products of three-point functions, five-point functions from four- and three-point functions and so on, ultimately all the amplitudes can be derived from the three point functions alone. For pure Yang-Mills, the non-vanishing three-point amplitudes are given by

$$M_3(g_1^+, g_2^+, g_3^-) = \sqrt{2}\mathrm{i}\frac{[21]^3}{[32]\,[13]}\ , \ M_3(g_1^+, g_2^-, g_3^-) = \sqrt{2}\mathrm{i}\frac{\langle 23\rangle^3}{\langle 12\rangle\,\langle 31\rangle}. \tag{9.21}$$

For real momenta, the three point amplitudes vanish for external on-shell momenta since e.g. $0 = k_3^2 = (k_1{+}k_2)^2 = \langle 12\rangle\,[21]$ and $[21] \sim \langle 21\rangle^*$ imply that all the spinor brakets vanish and there are more brakets in the numerator than in the denominator. For the on-shell recursion relation we need the three-point vertices evaluated with the shifted spinors (9.7) in which case the amplitudes can be non-vanishing.

The application of the on-shell recursion relations is simplified by the following observation:

- the three-point vertex $M_3(g^+, g^+, g^-)$ vanishes if it involves the gluon j.
- the three-point vertex $M_3(g^-, g^-, g^+)$ vanishes if it involves the gluon i.

This can seen by the following argument. We discuss the case of leg j explicitly, the result for leg i can be seen analogously. Consider the anti-MHV type three-point functions with particle j and two positive helicity gluons:

$$M_3(g_j^{'-}, g_k^+, g_l^+) \sim \frac{[kl]^3}{[j'k]\,[lj']} \quad M_3(g_j^{'+}, g_k^+, g_l^-) \sim \frac{[j'k]^3}{[kl]\,[lj']} \tag{9.22}$$

We assume the kinematics to be such that for undeformed momenta $2(k_j \cdot k_k) = \langle jk\rangle\,[kj] \neq 0$. For the deformed kinematics we have instead $0 = k_l^2 = 2(k_j' \cdot k_k) = \langle jk\rangle\,[kj']$. Since $\langle jk\rangle \neq 0$ by assumption this means that $[kj'] = 0$ so that the second vertex above vanishes. With a similar argument one finds that $[lj'] = 0$. But this implies[2] that

$$|j'] \propto |k] \propto |l] \tag{9.23}$$

so that all the scalar products vanish. As a consequence, also the first vertex above is zero.

For the MHV-type three-point vertices with leg j we have

$$M_3(g_j^{'+}, g_k^-, g_l^-) \sim \frac{\langle kl\rangle^3}{\langle jk\rangle\,\langle lj\rangle} \quad M_3(g_j^{'-}, g_k^-, g_l^+) \sim \frac{\langle jk\rangle^3}{\langle kl\rangle\,\langle lj\rangle} \tag{9.24}$$

since $|j'\rangle = |j\rangle$. These vertices are not modified by the shift and therefore are nonvanishing.

Derivation of three-gluon amplitude We give the explicit derivation of one of the three-point functions. The usual three-gluon vertex function is

$$V_{3,\mu\nu\lambda}(p_1, p_2, -(p_1+p_2)) = \mathrm{i}\,[g_{\nu\lambda}(2p_2+p_1)_\mu - g_{\lambda\mu}(2p_1+p_2)_\nu + g_{\mu\nu}(p_1-p_2)_\lambda] \tag{9.25}$$

The contraction with one negative-helicity polarization vector and two positive-helicity polarization vectors with the same reference spinor is

$$\begin{aligned}
V_3^{\mu\nu\lambda}\epsilon_\mu^{+*}(p_1,q)\epsilon_\mu^{+*}(p_2,q)\epsilon_\mu^{-*}(p_3) &= \frac{\mathrm{i}}{\sqrt{2}\,\langle q1\rangle\,\langle q2\rangle\,[3q]}\Big[\langle q|\,\gamma^\lambda|2]\,\langle q|\,\not{p}_2|1] - \langle q|\,\gamma^\lambda|1]\,\langle q|\,\not{p}_1|2]\Big]\,[q|\gamma_\lambda\,|3\rangle \\
&= \frac{\mathrm{i}\,[21]}{\sqrt{2}\,\langle q1\rangle\,\langle q2\rangle\,[3q]}\Big[\langle q|\,\gamma^\lambda|2]\,\langle q2\rangle + \langle q|\,\gamma^\lambda|1]\,\langle q1\rangle\Big]\,[q|\gamma_\lambda\,|3\rangle \\
&= \sqrt{2}\mathrm{i}\frac{[21]\,\langle 3q\rangle}{\langle q1\rangle\,\langle q2\rangle\,[3q]}\Big[\underbrace{\langle q2\rangle\,[2q] + \langle q1\rangle\,[1q]}_{=\langle q|\not{k}_{1,2}|q]=-\langle q3\rangle[3q]}\Big] \\
&= -\sqrt{2}\mathrm{i}\frac{[12]\,\langle 3q\rangle^2}{\langle q1\rangle\,\langle q2\rangle} = \sqrt{2}\mathrm{i}\frac{[21]^4}{[32]\,[21]\,[13]}
\end{aligned} \tag{9.26}$$

where momentum conservation has been used in the last step, e.g. $\langle q3\rangle\,[32] = -\langle q1\rangle\,[12]$.

[2] It is a property of the vector space of the two-dimensional spinors that two spinors with $[ij] = 0$ are proportional, as can be seen e.g. from the Schouten identity: $|i]\,[jk] + |j]\,[ki] = 0$.

9.2.2 4-point amplitudes from recursion

As a first example for the application of on-shell recursion, consider the 4-gluon amplitude $M(g_1^+, g_2^+, g_3^-, g_4^-)$. We perform a shift (9.7) with $i = 1$ and $j = 4$ which leads to a valid recursion according to the discussion in Section 9.1.3. Since we consider colour-ordered amplitudes and legs one and four have to be in different sub-amplitudes, there is only one topology contributing to the recursion:

$$\mathrm{i}M_4(g_1^+, g_2^+, g_3^-, g_4^-) = \mathrm{i}M_3(g_1^{'+}, g_2^+, g_{-k'_{1,2}}^-)\frac{\mathrm{i}}{k_{1,2}^2}\mathrm{i}M_3(g_{k'_{1,2}}^+, g_3^-, g_4^{'-}) \tag{9.27}$$

The sum over the internal helicities has collapsed since there is no three-point vertex with only positive helicity gluons. The left three-point vertex is

$$\mathrm{i}M_3(g_1^{'+}, g_2^+, g_{-k'_{1,2}}^-) = \sqrt{2}\mathrm{i}\frac{[21']^3}{[-k'_{1,2}2]\,[1'(-k'_{1,2})]} = -\sqrt{2}\mathrm{i}\frac{[21]^3}{[k'_{1,2}2]\,[1k'_{1,2}]} \tag{9.28}$$

Here it was used that only the spinors $|1\rangle$ are shifted in (9.7) and we use spinor conventions where $|-(K)\pm\rangle = i\,|K\pm\rangle$. The right three-point vertex is

$$M_3(g_{k'_{1,2}}^+, g_3^-, g_4^{'-}) = \sqrt{2}\mathrm{i}\frac{\langle 34\rangle^3}{\langle k'_{1,2}3\rangle\,\langle 4k'_{1,2}\rangle} \tag{9.29}$$

where it was used that only $|4]$ is shifted. There is now a standard trick to be used to simplify the spinor products involving $k'_{1,2}$. This is based on the observation that the shift drops out if

$$\not{k}'_{1,2} = \not{k}_{1,2} + z\eta = \not{k}_{1,2} + \frac{z}{2}(|1]\,\langle 4| + |4\rangle\,[1|) \tag{9.30}$$

is sandwiched between $|1]$ or $|4\rangle$. Therefore multiplying and dividing by appropriate spinor products one can write

$$\langle Ak'_{1,2}\rangle\,[k'_{1,2}B] = \frac{\langle Ak'_{1,2}\rangle\,[k'_{1,2}1]\,\langle 4k'_{1,2}\rangle\,[k'_{1,2}B]}{\langle 4k'_{1,2}\rangle\,[k'_{1,2}1]} = \frac{\langle A|\,\not{k}_{1,2}|1]\,\langle 4|\,\not{k}_{1,2}|B]}{\langle 4|\,\not{k}_{1,2}|1]} \tag{9.31}$$

Applying this trick to the spinor products in the denominator and using identities such as $\langle 4|\,\not{k}_{1,2}|1] = \langle 42\rangle\,[21]$ gives the amplitude in the simple form

$$\begin{aligned}\mathrm{i}M_4(g_1^+, g_2^+, g_3^-, g_4^-) &= (-\mathrm{i})(\sqrt{2}\mathrm{i})^2\frac{[21]^3}{\langle 4|\,\not{k}_{1,2}|2][1|\not{k}_{1,2}\,|4\rangle}\frac{[1|\not{k}_{1,2}\,|4\rangle^2}{2(k_1\cdot k_2)}\frac{\langle 34\rangle^3}{[1|\not{k}_{1,2}\,|3\rangle\,\langle 4|\,\not{k}_{1,2}|1]}\\ &= 2\mathrm{i}\frac{[21]^3}{\langle 41\rangle\,[12]}\frac{1}{[12]\,\langle 21\rangle}\frac{\langle 34\rangle^3}{[12]\,\langle 23\rangle}\\ &= 2\mathrm{i}\frac{\langle 34\rangle^3}{\langle 12\rangle\,\langle 23\rangle\,\langle 41\rangle}\end{aligned} \tag{9.32}$$

in agreement with the previous Feynman-diagram calculation.

9.2.3 MHV amplitudes

The Parke-Taylor formula for the MHV amplitudes (8.30) can now be proven easily by induction using the on-shell recursion relations [12]. Without loss of generality we can always consider gluon n to have negative helicity. Similar to the four-point example we use the shift (9.7) for $(i,j) = (1,n)$. Several contributions to the recursion vanish since there are no amplitudes with only one negative leg, apart from the three-point functions. The two potentially non-vanishing contributions are therefore

$$\begin{aligned} M_n(g_1^+, \ldots g_l^-, \ldots, g_n^-) =& M_3(g_1^{\prime +}, g_2^+, g^-_{-k'_{1,2}}) \frac{\mathrm{i}}{k^2_{1,2}} M_{n-1}(g^+_{k'_{1,2}}, \ldots g_l^-, \ldots, g_n^{\prime -}) \\ &+ M_{n-1}(g_1^{\prime +}, \ldots g_l^-, \ldots, g^+_{n-2}, g^-_{-k'_{1,n-2}}) \frac{\mathrm{i}}{k^2_{1,n-2}} M_3(g^{\prime +}_{k_{1,n-2}}, g^+_{n-1}, g_n^{\prime -}) \end{aligned} \tag{9.33}$$

Here we consider the case where $l \neq 2$ and $l \neq n-1$. The three point-vertex in the second line vanishes for the reasons discussed in Section 9.2.1. The only contribution is thus given by

$$\begin{aligned} M_n(g_1^+, \ldots g_l^-, \ldots, g_n^-) &= M_3(g_1^{\prime +}, g_2^+, g^{\prime -}_{-k_{1,2}}) \frac{\mathrm{i}}{k^2_{1,2}} M_{n-1}(g^{\prime +}_{k_{1,2}}, \ldots g_l^-, \ldots g_n^{\prime -}) \\ &= \mathrm{i} 2^{n/2-1} \frac{[21]^3}{[k'_{1,2}2]\,[1k'_{1,2}]} \frac{1}{\langle 12\rangle\,[21]} \frac{\langle ln\rangle^4}{\langle k'_{1,2}3\rangle\,\langle 34\rangle \ldots \langle (n-1)n\rangle\,\langle nk'_{1,2}\rangle} \\ &= \mathrm{i} 2^{n/2-1} \frac{[12]^2}{\langle 12\rangle\,\langle n|\,\not{k}_1|2][1|\not{k}_2\,|3\rangle} \frac{\langle ln\rangle^4}{\langle 34\rangle \ldots \langle (n-1)n\rangle} \\ &= \mathrm{i} 2^{n/2-1} \frac{\langle ln\rangle^4}{\langle 12\rangle\,\langle 23\rangle \ldots \langle n1\rangle} \end{aligned}$$

so that the MHV formula is proven by induction.

For more examples and tricks on the application of the on-shell recursion relations see e.g. the original paper [12] and the lecture notes [9].

Part III

NLO calculations in QCD

Chapter 10

NLO methods: the example $e^+e^- \to$ Hadrons

10.1 NLO contributions to $e^+e^- \to$ Hadrons

Recall the expression for the R-ratio as an expectation value of electromagnetic quark currents:

$$
\begin{aligned}
R(q^2) &= \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma^0(e^+e^- \to \mu^+\mu^-)} \\
&= -\frac{2\pi}{q^2}\int \mathrm{d}^4x e^{-\mathrm{i}q\cdot x} \langle 0|j_q^\mu(x) j_{q,\mu}(0)|0\rangle \\
&= -\frac{2\pi}{q^2}\int \mathrm{d}^4x e^{-\mathrm{i}q\cdot x} \sum_x \int \mathrm{d}\Phi_x \langle 0|j_q^\mu(x)|x\rangle \langle x|j_{q,\mu}(0)|0\rangle
\end{aligned}
\tag{10.1}
$$

The sum is over a complete set of **partonic** final states

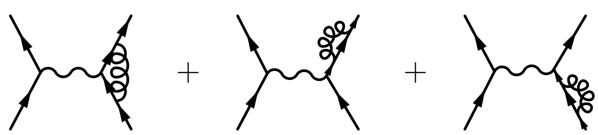
$$\{|x\rangle\} = \{|q\bar{q}\rangle, |q\bar{q}g\rangle, |q\bar{q}gg\rangle, |q\bar{q}q\bar{q}\rangle, \dots\} \tag{10.2}$$

The LO prediction results from taking the only $q\bar{q}$ states into account, which amounts to using the LO cross section

$$\sigma(e^+e^- \to q\bar{q}) = N_c Q_q^2 \frac{4\pi\alpha^2}{3s}, \tag{10.3}$$

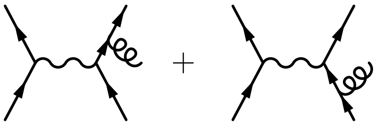
summed over quark flavours, in the numerator of the R-ratio.

The first contribution to the QCD corrections to the R-ratio is given by the $\mathcal{O}(\alpha_s)$ corrections that arise from one-loop corrections $e^-e^+ \to q\bar{q}$,



(10.4)

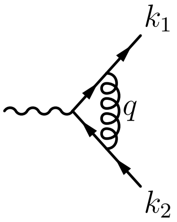
and the tree contribution of the $q\bar{q}g$ term in the sum over partonic final states,



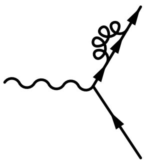
$$+ \qquad (10.5)$$

10.2 Regularization and renormalization

The one-loop corrections to $e^-e^+ \to q\bar{q}$ in (10.4) involve the one-loop subdiagrams



$$= (\mathrm{i})^2(-\mathrm{i}eQ_q)(\mathrm{i}g_s)^2 C_F \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\bar{u}(k_1)\gamma^\nu(\not{q} - \not{k}_1)\gamma^\mu(\not{k}_2 + \not{q})\gamma_\nu v(k_2)}{((k_2+q)^2 + \mathrm{i}\epsilon)(q^2+\mathrm{i}\epsilon)((k_1-q)^2+\mathrm{i}\epsilon)} \qquad (10.6)$$

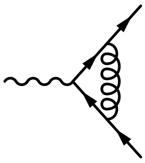


$$= (\mathrm{i})^2(-\mathrm{i}eQ_q)(\mathrm{i}g_s)^2 C_F \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\bar{u}(k_1)\gamma^\nu(\not{q} - \not{k}_1)\gamma_\nu \not{k}_1 \gamma^\mu v(k_2)}{((k_1-q)^2 + \mathrm{i}\epsilon)(q^2+\mathrm{i}\epsilon)(k_1^2+\mathrm{i}\epsilon)} \qquad (10.7)$$

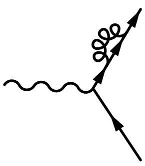
We will next discuss conceptual and technical issues arising in the computation of such loop integrals before returning to the example of $e^-e^+ \to$ Hadrons.

UV divergences

Since the integration runs over all of momentum space, the loop integrals also receive contributions from $q \to \infty$. Introducing a cutoff Λ, the integrals (10.6) and (10.7) behave schematically as



$$\sim \int^\Lambda \mathrm{d}^4 q \frac{q^2}{q^6} \sim \log\Lambda \qquad (10.8)$$



$$\sim \int^\Lambda \mathrm{d}^4 q \frac{q}{q^4} \sim \Lambda \qquad (10.9)$$

Therefore these integrals diverge in the limit $\Lambda \to \infty$.

In order to make sense out of these integrals one introduces the steps of **regularization** and **renormalization**.

10.2.1 Regularization

The theory is regularized by modifying it introducing regulator so that all loop integrals are defined and can be calculated. Examples for regulators are

- An explicit momentum cutoff, $k < \Lambda$
- Lattice regulator: space-time is discretized
- Pauli-Villars regularization: propagators are replaced by

$$\frac{\mathrm{i}}{p^2 - m^2} \to \frac{\mathrm{i}}{p^2 - m^2} - \frac{\mathrm{i}}{p^2 - M^2}$$

  with a large regulator mass M.

- Subtraction: introduce a unique prescription to subtract the divergent parts of the integrand
- Dimensional regularization: change the dimension from 4 to $d \neq 4$ so that the integrals converge.

The drawback of the cutoff and lattice prescriptions is that they break Lorentz invariance, Pauli-Villars regularization has been used in QED but breaks gauge invariance in QCD. Subtraction prescriptions have been used in renormalizability proofs but are tedious for practical calculations.

Dimensional regularization

For most practical calculations in perturbation theory, **dimensional regularization** is used. The loop integral measure is changed to

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \to \mu^{4-d} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \tag{10.10}$$

where μ is an arbitrary constant with dimension of mass so that the dimension of integrals is not changed.

In order to discuss the limit $d \to 4$ we write

$$d = 4 - 2\varepsilon \tag{10.11}$$

and consider loop integrals as analytic functions in ε. (Here it is hoped that no confusion arises with the $\mathrm{i}\epsilon$ prescription of the Feynman propagators).

10.2.2 Renormalization

The values of the parameters of a QFT (masses, coupling constants) have to be extracted from a set of observables $\mathcal{O}_i$. The parameters in the original Lagrangian of the theory will from now on be called **bare parameters** and denoted as g_0, m_0 etc. In the regularized theory, the relations expressing an observable $\mathcal{O}$ computed in terms of the bare parameters depends on the regularization parameters, in case of dimensional regularization on the scale μ and the dimension d:

$$\mathcal{O}(g_0, \mu, d) \tag{10.12}$$

The idea of renormalization is to introduce **renormalized** fields, masses and coupling constants Φ_r, m_r, g_r so that the expressions of observables in terms of renormalized quantities is independent of the regularization. QFTs where this is possible are called **renormalizable**.

The bare quantities are related to the renormalized ones by introducing **renormalization constants** Z:

$$\begin{aligned} \Phi_0 &= \sqrt{Z_\Phi}\Phi_r \\ g_0 &= Z_g g_r \\ m_0 &= Z_m m_r \end{aligned} \tag{10.13}$$

The renormalization constants are functions of the renormalized parameters of the theory and the regularization parameters:

$$Z = Z(g_r, m_r, \mu, d) \tag{10.14}$$

The Lagrangian of the theory in terms of the renormalized quantities is obtained by inserting the relations (10.13):

$$\mathcal{L}(\Phi_0, g_0, m_0) = \mathcal{L}(Z_\Phi^{1/2}\Phi_r, Z_g g_r, Z_m m_r) \tag{10.15}$$

For the evaluation in perturbation theory, the renormalization constants are written as

$$Z = 1 + \delta Z \tag{10.16}$$

where δZ is of order g_r. The Lagrangian becomes then

$$\mathcal{L}(\Phi_0, g_0, m_0) = \mathcal{L}(\Phi_r, g_r, m_r) + \delta\mathcal{L}(\Phi_r, g_r, m_r, \delta Z_\Phi^{1/2}, \delta Z_g, \delta Z_m) \tag{10.17}$$

The first term is the original Lagrangian with all fields and parameters replaced by the renormalized quantities. The second term $\delta\mathcal{L}$ is called the **counterterm Lagrangian**. As example consider a fermion interacting with a vector boson with the Lagrangian

$$\mathcal{L}(\psi_0, A_0, g_0, m_0) = \bar{\psi}_0 \left(\mathrm{i}\partial\!\!\!/ - m_0\right) \psi_0 + g_0\bar{\psi}_0 A\!\!\!/_0\psi_0 \tag{10.18}$$

where for simplicity we do not show the kinetic Lagrangian for the vector boson. Inserting the renormalization transformation, the Lagrangian becomes

$$\mathcal{L}(\psi_0, A_0, g_0, m_0) = \mathcal{L}(\psi_r, A_r, g_r, m_r) + \delta Z_\psi \bar{\psi}_r \mathrm{i}\partial\!\!\!/\psi_r - \delta m\bar{\psi}_r\psi_r + \delta g\bar{\psi}_r A\!\!\!/_r\psi_r \tag{10.19}$$

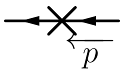
with the **mass counterterm**

$$\delta m = m_r(Z_\psi Z_m - 1) = m_r(\delta Z_\psi + \delta Z_m + \dots) \tag{10.20}$$

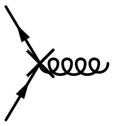
and the **coupling counterterm**

$$\delta g = g_r(Z_\psi Z_g Z_A^{1/2} - 1) = g_r(\delta Z_\psi + \delta Z_g + \frac{1}{2}\delta Z_A + \dots) \equiv g_r \delta Z_{\psi\psi A} \tag{10.21}$$

The terms in the counterterm Lagrangian are treated perturbatively as additional interactions with the Feynman rules



$$: \mathrm{i}\delta Z_\psi \not{p} - \mathrm{i}\delta m = \mathrm{i}\delta Z_\psi(\not{p} - m_r) - \mathrm{i}m_r\delta Z_m, \tag{10.22}$$



$$: -\mathrm{i}g_r \delta Z_{\psi\psi A}\gamma^\mu \,, \tag{10.23}$$

Mass dimensions in dimensional regularization

Note that in dimensional regularization the action is defined as

$$S = \int \mathrm{d}^d x \mathcal{L} \tag{10.24}$$

In the conventions where $\hbar c = 1$ the action is dimensionless, so the Lagrangian has mass dimension $d = 4 - 2\varepsilon$. This implies that the mass dimension of the quark and gluon fields is

$$\begin{aligned} [\psi] &= \frac{d-1}{2} = \frac{3}{2} + \varepsilon \\ [A] &= \frac{d-2}{2} = 1 + \varepsilon \end{aligned} \tag{10.25}$$

Since the interaction term $\sim g_0 \bar{\psi}_0 A_0 \psi_0$ has dimension d, the bare coupling constant is not dimensionless in d dimensions:

$$[g_0] = d - \left((d-1) + \frac{d-2}{2}\right) = -\frac{d-4}{2} = \varepsilon \tag{10.26}$$

We absorb this mass dimension by defining the dimensionless, renormalized coupling g as

$$g = \mu^{-\varepsilon} g_r = \mu^{-\varepsilon} Z_g^{-1} g_0 \tag{10.27}$$

This definition is consistent with the fact that each loop integral appears with a factor g_r^2 so that every loop diagram involves the combination

$$g_r^2 \int \frac{\mathrm{d}^d q}{(2\pi)^d} = g^2 \mu^{2\varepsilon} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \tag{10.28}$$

in agreement with our previous discussion in eq. (10.10).

10.3 One-Loop integrals

Instead of attacking directly the computation of one-loop integrals such as (10.6) we will consider so-called **scalar one-loop integrals** where the numerator structure is trivial.

Scalar two- and three-point integrals are defined as

$$B_0(p^2, m_1, m_2) = \frac{(2\pi\mu)^{4-d}}{\mathrm{i}\pi^2} \int \mathrm{d}^d q \frac{1}{(q^2 - m_1^2)((q+p)^2 - m_2^2)}, \tag{10.29}$$

$$C_0(k_1^2, k_2^2, k_3^2, m_1, m_2, m_3) = \frac{(2\pi\mu)^{4-d}}{\mathrm{i}\pi^2} \int \mathrm{d}^d q \frac{1}{(q^2 - m_1^2)((q+k_1)^2 - m_2^2)((q+k_1+k_2)^2 - m_3^2)}, \tag{10.30}$$

where all momenta are incoming and momentum conservation holds, i.e. $p_1 + p_2 + p_3 = 0$ etc.

The prefactor in the scalar integrals has been extracted by convention, the relation to the loop-integral measure is

$$\frac{(2\pi\mu)^{4-d}}{\mathrm{i}\pi^2} \mathrm{d}^d q = (-\mathrm{i})(4\pi)^2 \mu^{4-d} \frac{\mathrm{d}^d q}{(2\pi)^d}. \tag{10.31}$$

10.3.1 Example calculation: scalar two-point function

We discuss some standard tricks in the computation of one-loop integrals for the example of a two-point function with massless propagators.

Feynman parameters The first step is to combine the two propagators at the cost of introducing another integral using the identity

$$\frac{1}{a_1 a_2} = \int_0^1 dx \frac{1}{(x a_1 + (1-x) a_2)^2}. \tag{10.32}$$

This results in the expression

$$\begin{aligned} \frac{\mathrm{i}}{(4\pi)^2} B_0(k^2, 0, 0) &= \mu^{4-d} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 + \mathrm{i}\epsilon)((q+k)^2 + \mathrm{i}\epsilon)} \\ &= \mu^{4-d} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{1}{(q^2 + (1-x)[k^2 + 2(k \cdot q)] + \mathrm{i}\epsilon)^2}. \end{aligned} \tag{10.33}$$

The parameter x is called a Feynman parameter.

Shift of loop momentum In the next step, a variable transformation

$$q = \ell - (1-x)k \tag{10.34}$$

is used to eliminate the $q \cdot k$ mixing term:

$$\frac{\mathrm{i}}{(4\pi)^2} B_0(k^2, 0, 0) = \mu^{4-d} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{1}{(\ell^2 + x(1-x)k^2 + \mathrm{i}\epsilon)^2}. \tag{10.35}$$

Wick rotation To evaluate the ℓ integral, one deforms the integration in the complex plane so that the ℓ^0 integral runs from $-\mathrm{i}\infty < \ell^0 < \mathrm{i}\infty$. This is possible under the assumption $k^2 < 0$ since the poles in ℓ^0,

$$\ell^0 = \pm\sqrt{\vec{\ell}^2 - x(1-x)k^2 - \mathrm{i}\epsilon}, \tag{10.36}$$

lie in the right-lower and left-upper half-planes. Redefining $\ell^0 \to \mathrm{i}\ell_E^0$ the Minkowski scalar product becomes an euclidean product $\ell^2 \Rightarrow -(\ell_E^0) - \vec{\ell}^2$. The resulting integral can then be written in d-dimensional spherical coordinates:

$$\int \mathrm{d}^d\ell_E = \int \mathrm{d}\Omega_d \int \mathrm{d}\ell_E \ell_E^{d-1} = \frac{1}{2}\int \mathrm{d}\Omega_d \int \mathrm{d}\ell_E^2 (\ell_E^2)^{(d-2)/2} \tag{10.37}$$

with the d-dimensional unit sphere

$$\int \mathrm{d}\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{10.38}$$

Integral over the loop momentum Performing the Wick rotation and introducing spherical coordinates, the integral over the loop momentum is reduced to a one-dimensional integral:

$$\begin{aligned}\frac{\mathrm{i}}{(4\pi)^2}B_0(k^2,0,0) =& \mathrm{i}\mu^{4-d}\int \frac{\mathrm{d}^d\ell_E}{(2\pi)^d}\int_0^1 \mathrm{d}x\,\frac{1}{(-\ell_E^2 + x(1-x)k^2 + \mathrm{i}\epsilon)^2} \\ =& \,\mathrm{i}\mu^{4-d}\frac{\pi^{d/2}}{(2\pi)^d\Gamma(d/2)}\int \mathrm{d}\ell_E^2 \frac{(\ell_E^2)^{(d-2)/2}}{(\ell_E^2 - x(1-x)k^2 - \mathrm{i}\epsilon)^2} \\ =& \,\frac{\mathrm{i}(\mu^2)^{2-d/2}\Gamma(2-d/2)}{(4\pi)^{d/2}}(-k^2 - \mathrm{i}\epsilon)^{d/2-2}\int_0^1 \mathrm{d}x\,(x(1-x))^{d/2-2}\end{aligned} \tag{10.39}$$

Here the ℓ_E integral has been performed using the formula

$$\int_0^\infty d\ell_E^2\,(\ell_E^2)^\alpha (A + \ell_E^2)^{-2} = (A)^\alpha \Gamma(\alpha+1)\Gamma(1-\alpha) \tag{10.40}$$

with the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt \tag{10.41}$$

For non-negative integers,

$$\Gamma(n+1) = n! \tag{10.42}$$

A useful property of the Gamma function is

$$\Gamma(z+1) = z\Gamma(z) \tag{10.43}$$

Feynman parameter integral The remaining integral over the Feynman parameter can be performed using the integral representation of the Beta function

$$\int_0^1 \mathrm{d}x\, x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{10.44}$$

Using this expression finally gives the result for the scalar two-loop function

$$\begin{aligned} B_0(k^2,0,0) &= (4\pi)^{2-d/2}\left(-\frac{k^2+\mathrm{i}\epsilon}{\mu^2}\right)^{d/2-2}\frac{\Gamma(2-d/2)\Gamma(d/2-1)^2}{\Gamma(d-2)} \\ &= (4\pi)^{\varepsilon}\left(-\frac{k^2+\mathrm{i}\epsilon}{\mu^2}\right)^{-\varepsilon}\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)} \end{aligned} \tag{10.45}$$

The complex continuation from $k^2 < 0$ to positive values is defined by the iϵ prescription. In the last step, the number of the dimensions has been set to $d = 4 - 2\varepsilon$.

Expansion in ε The Gamma function has poles for $x = 0, -1, -2, \ldots$. The pole in the two-point function for $\varepsilon \to 0$ reflects the original UV singularity in the unregularized integral. The d-dimensional result can be expanded around $d = 4$ using the series expansion of the Gamma function

$$\Gamma(x) = \frac{1}{x} - \gamma_E + \frac{1}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right)x + \mathcal{O}(x^2) \tag{10.46}$$

with the Euler-Mascheroni constant

$$\gamma_E = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k} - \ln n\right) = 0.57721\ldots, \tag{10.47}$$

and the expansion

$$A^x = e^{x\ln A} = 1 + x\ln A + \ldots. \tag{10.48}$$

For the two-point function, one obtains the result

$$B_0(k^2,0,0) = \left[\frac{1}{\varepsilon} - \left(-\log\left(\frac{k^2+\mathrm{i}\epsilon}{\mu^2}\right) - 2\right) - \gamma_E + \log(4\pi)\right] + \mathcal{O}(\varepsilon) \tag{10.49}$$

Since the $\log(4\pi) - \gamma_E$ terms are usually absorbed in the renormalization condition, one often extracts a prefactor, for instance

$$(4\pi)^{\varepsilon}\Gamma(1+\varepsilon) = 1 + \varepsilon(\log(4\pi) - \gamma_E) + \ldots \tag{10.50}$$

so that

$$B_0(k^2,0,0) = \left[\frac{(4\pi)^{\varepsilon}\Gamma(1+\varepsilon)}{\varepsilon} - \left(\log\left(\frac{-k^2-\mathrm{i}\epsilon}{\mu^2}\right) - 2\right)\right] + \mathcal{O}(\varepsilon) \tag{10.51}$$

For $k^2 > 0$ the iϵ prescription implies that the logarithm is interpreted as

$$\log\left(\frac{-k^2-\mathrm{i}\epsilon}{\mu^2}\right) \to \log\left(\frac{k^2}{\mu^2}\right) - \mathrm{i}\pi \tag{10.52}$$

10.3.2 One-loop scalar functions

Useful formulae

The above steps can be performed for any scalar one-loop integral. Feynman parameters can be introduced with the general formula:

$$\frac{1}{a_1^{m_1}\dots a_n^{m_n}} = \frac{\Gamma(m)}{\Gamma(m_1)\dots\Gamma(m_n)}\int_0^1 \mathrm{d}x_1 x_1^{m_1-1}\dots\int_0^1 \mathrm{d}x_n x_n^{m_n-1}\frac{\delta(1-\sum_i x_i)}{(x_1a_1+\dots+x_na_n)^m} \tag{10.53}$$

with $m=\sum_i m_i$. After shifting the loop momentum, the loop integral can be performed with the formula

$$\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{(\ell^2)^\alpha}{(\ell^2-M^2)^\beta} = (-)^{\alpha+\beta}\frac{\mathrm{i}}{(4\pi)^{d/2}}(M^2)^{\alpha-\beta+d/2}\frac{\Gamma(\alpha+d/2)\Gamma(\beta-\alpha-d/2)}{\Gamma(d/2)\Gamma(\beta)} \tag{10.54}$$

Note that this formula is valid for the loop momentum ℓ in Minkowski space.

Results for all the one-loop scalar integrals are available in the literature and implemented in computer libraries. For one-loop integrals, after expansion around $d=4$, the result of the Feynman-parameter integration can be written in terms of logarithms and dilogarithms, which are defined as

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\mathrm{d}z}{z}\ln(1-z) \tag{10.55}$$

One and three-point functions

We give here two more results that arise in the calculation of $e^-e^+\to$ hadrons. The simplest loop function is the scalar one-point integral

$$\begin{aligned}
A_0(m) &= \frac{(2\pi\mu)^{4-d}}{\mathrm{i}\pi^2}\int \mathrm{d}^d q\frac{1}{(q^2-m^2)}\\
&= -(4\pi)^\varepsilon\Gamma(\varepsilon-1)m^2\left(\frac{m^2-\mathrm{i}\varepsilon}{\mu^2}\right)^{-\varepsilon}\\
&= (4\pi)^\varepsilon\Gamma(\varepsilon+1)m^2\left[\frac{1}{\varepsilon}-\ln\left(\frac{m^2}{\mu^2}\right)+1+\mathcal{O}(\varepsilon)\right]
\end{aligned} \tag{10.56}$$

For $m=0$ this integral is defined to be zero, which corresponds to taking the limit $m\to 0$ in the regularized integral before expanding in ε and assuming $\varepsilon<0$.

As another example, the scalar one-loop integral with vanishing internal masses and one off-shell leg is given by

$$\begin{aligned}
C_0(k^2,0,0,0,0) &= (4\pi)^\varepsilon\Gamma(1+\varepsilon)\mu^{2\varepsilon}(-k^2-\mathrm{i}\varepsilon)^{-1-\varepsilon}\frac{1}{\varepsilon}\frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\\
&= (4\pi)^\varepsilon\Gamma(1+\varepsilon)\mu^{2\varepsilon}(-k^2-\mathrm{i}\varepsilon)^{-1-\varepsilon}\left(-\frac{1}{\varepsilon^2}+\frac{\pi^2}{6}\right)\\
&= (4\pi)^\varepsilon\Gamma(1+\varepsilon)\frac{1}{k^2}\left[\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\log\left(-\frac{k^2}{\mu^2}\right)+\frac{1}{2}\log^2\left(-\frac{k^2}{\mu^2}\right)-\frac{\pi^2}{6}\right]
\end{aligned} \tag{10.57}$$

Note that the scalar one-loop integral behaves for large loop momentum as

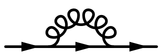
$$\int^\Lambda \mathrm{d}^4 q \frac{1}{q^6} \tag{10.58}$$

so it is UV finite. The $\frac{1}{\varepsilon^2}$ singularity in the result is instead related to a so-called **infrared singularity** that will be discussed below.

10.3.3 Self-energy and vertex functions

Quark self energy

We discuss the steps involved in calculating loop diagrams for the example of the quark self energy

$\Sigma(k^2) =$ 

$$= \mathrm{i} g_s^2 C_F \mu^{4-d} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\gamma^\nu(\not{q} + \not{k})\gamma_\nu}{((k+q)^2 + \mathrm{i}\epsilon)(q^2 + \mathrm{i}\epsilon)} \tag{10.59}$$

The steps appearing here are the simplest example for methods used in general in one loop calculations. We will only consider Feynman gauge, $\xi = 1$, which has been used already in the above expression. The quantity g_s is the dimensionless renormalized coupling defined as in (10.27).

***d*-dimensional Dirac algebra** In dimensional regularization, also the Dirac and Tensor algebra is performed in d-dimensions. The trace of the metric tensor in d dimensions is

$$g^\mu_\mu = d. \tag{10.60}$$

This definition leads e.g. to the identities

$$\gamma^\mu \gamma_\mu = g^\mu_\mu = d, \tag{10.61}$$
$$\gamma^\nu \gamma^\mu \gamma_\nu = (2-d)\gamma^\mu, \tag{10.62}$$
$$\gamma^\nu \gamma^\mu \gamma^\rho \gamma_\nu = 4g^{\mu\nu} + (d-4)\gamma^\mu\gamma^\rho \tag{10.63}$$
$$\gamma^\nu \gamma^\mu \gamma^\rho \gamma^\sigma \gamma_\nu, = -2\gamma^\sigma\gamma^\rho\gamma^\mu + (4-d)\gamma^\mu\gamma^\rho\gamma^\sigma. \tag{10.64}$$

The unit matrix in the space of Dirac spinors can be taken to satisfy

$$\mathrm{tr}[\mathbf{1}] = 4, \tag{10.65}$$

so that the equations for traces of Dirac matrices remain valid.

Using the identity (10.62) the integral appearing in the self-energy (10.7) becomes

$$\mathrm{i}\mu^{4-d} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\gamma^\nu(\not{q} + \not{k})\gamma_\nu}{((k+q)^2 + \mathrm{i}\epsilon)(q^2 + \mathrm{i}\epsilon)} = (2-d)\frac{1}{(4\pi)^2}\gamma^\mu(k_\mu B_0(k^2,0,0) + B_\mu(k^2,0,0)) \tag{10.66}$$

Here the **vector integral**

$$B^\mu(p^2, m_1, m_2) = \frac{(2\pi\mu)^{4-d}}{\mathrm{i}\pi^2} \int \mathrm{d}^d q \frac{q^\mu}{(q^2 - m_1^2)((q+p)^2 - m_2^2)} \tag{10.67}$$

was introduced.

Tensor reduction Instead of computing vector integrals (and analogous **tensor integrals** with insertions of several loop momenta in the numerator) directly, a useful strategy is to express them as linear combination of scalar integrals. This can be done for all one-loop integrals and is known as **tensor reduction** or Passarino-Veltman reduction. For the two-point vector integral we note that Lorentz invariance implies that it can be written in the form

$$B^\mu(p^2, m_1, m_2) = p^\mu B_1(p^2, m_1, m_2) \tag{10.68}$$

with the **vector coefficient** B_1. This expression can be solved for the vector coefficient:

$$B_1(p^2, m_1, m_2) = \frac{1}{p^2} p_\mu B^\mu(p^2, m_1, m_2) = \frac{(2\pi\mu)^{4-d}}{\mathrm{i}\pi^2} \int \mathrm{d}^d q \frac{p \cdot q}{(q^2 - m_1^2)((q+p)^2 - m_2^2)} \tag{10.69}$$

The basic idea of tensor reduction is to use the identity

$$(p \cdot q) = \frac{1}{2} \left[((q+p)^2 - m_2^2) - (q^2 - m_1^2) - (p^2 + m_1^2 - m_2^2) \right] \tag{10.70}$$

to eliminate the scalar product in the numerator and to cancel one of the propagators. This allows to express the vector integral in terms of scalar one and two-point integrals

$$B_1(p^2, m_1, m_2) = \frac{1}{2p^2} p_\mu \left[-(p^2 + m_1^2 - m_2^2) B_0(p^2, m_1, m_2) + A_0(m_1) - A_0(m_2) \right] \tag{10.71}$$

For the massless case, we obtain the result needed for the quark self energy:

$$B^\mu(k^2, 0, 0) = -\frac{1}{2} k^\mu B_0(k^2, 0, 0) \tag{10.72}$$

Result for the self energy Inserting the result of the vector integral, the self energy becomes

$$\begin{aligned} \Sigma(k^2) &= \frac{g_s^2}{(4\pi)^2} C_F 2(1-\varepsilon) \frac{1}{2} \not{k}\, B_0(k^2, 0, 0) \\ &= \frac{\alpha_s}{4\pi} C_F \not{k} \left[\frac{1}{\varepsilon} - \left(\log\left(\frac{-k^2}{\mu^2} \right) - 1 \right) - \gamma_E + \log(4\pi) \right] \end{aligned} \tag{10.73}$$

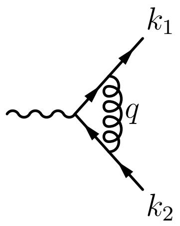
with

$$\alpha_s = \frac{g_s^2}{4\pi} \tag{10.74}$$

Note that as for the scalar one-point function, by definition we have $\Sigma(0) = 0$ in dimensional regularization.

Vertex function

The vertex function appearing in the corrections to $e^-e^+ \to q\bar{q}$ is given by

$\mathrm{i}\Gamma^\mu =$ 

$$=(-\mathrm{i}e)Q_q\,\mathrm{i}g_s^2 C_F \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\bar{u}(k_1)\gamma^\nu(\not{q} - \not{k}_1)\gamma_\nu \not{k}_1 \gamma^\mu v(k_2)}{((k_1-q)^2+\mathrm{i}\epsilon)(q^2+\mathrm{i}\epsilon)(k_1^2+\mathrm{i}\epsilon)} \tag{10.75}$$

After a somewhat tedious calculation using the Dirac equation for the spinors, the vertex corrections can be written in terms of the scalar two and three-point functions

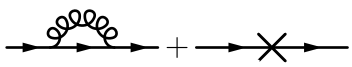
$$\begin{aligned}\Gamma^\mu &= (-\mathrm{i}eQ_q)(\bar{u}(k_1)\gamma^\mu v(k_2))\frac{\alpha_s}{4\pi}C_F[(d-7)B_0(k^2,0,0) - 2k^2 C_0(k^2,0,0,0,0,0)] \\ &= (-\mathrm{i}eQ_q)(\bar{u}(k_1)\gamma^\mu v(k_2))\frac{\alpha_s}{4\pi}C_F(4\pi)^\varepsilon\Gamma(1+\varepsilon)\left(-\frac{k^2+\mathrm{i}\epsilon}{\mu^2}\right)^{-\varepsilon}\left[\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+8+\frac{\pi^2}{3}\right].\end{aligned} \tag{10.76}$$

10.3.4 Renormalization constants of QCD

We here give the results for renormalization constants of QCD. We discuss the case of the quark field renormalization in some detail and introduce the modified minimal subtraction renormalization condition. The results for the remaining renormalization constants are quoted from the literature (e.g. [1, 7, 22]). We only consider Feynman gauge, $\xi = 1$.

Quark field renormalization

The **renormalized** quark self energy is given by the sum of the loop diagram and the counterterm to the propagator

$\Sigma_{\psi,r}(k^2) =$  $= \Sigma_\psi(k^2) + \delta Z_\psi \not{k}$ (10.77)

The renormalization constant Z_ψ is chosen such that the renormalized self energy is finite in the limit $\varepsilon \to 0$. This requirement fixes the singular part, but a possible finite part of δZ_ψ is not fixed. This requires to introduce some **renormalization condition**. Some possible renormalization conditions are

- Minimal subtraction (MS): only subtract the singular piece:

$$\delta Z_\psi^{(\mathrm{MS})} = -\frac{\alpha_s}{4\pi}C_F\frac{1}{\varepsilon} \tag{10.78}$$

- Modified minimal subtraction ($\overline{\text{MS}}$): subtract also the universal $\log(4\pi) - \gamma_E$ terms:

$$\delta Z_\psi^{\overline{(\text{MS})}} = -\frac{\alpha_s}{4\pi} C_F \underbrace{\left(\frac{1}{\varepsilon} + \log(4\pi) - \gamma_E\right)}_{\equiv \Delta_\varepsilon} \tag{10.79}$$

- Momentum subtraction (MOM): impose the condition $\Sigma_r(-\mu_0^2) = 0$ with some arbitrary momentum scale μ_0:

$$\delta Z_\psi^{(\text{MOM})}(\mu_0) = -\frac{\alpha_s}{4\pi} C_F \left[\frac{1}{\varepsilon} - \left(\log\left(\frac{\mu_0^2}{\mu^2}\right) - 1\right) - \gamma_E + \log(4\pi)\right] \tag{10.80}$$

- On-shell renormalization: for massive fermions the renormalized self energy has the structure

$$\Sigma_r(k^2) = \not{k}\,\Sigma_V(k^2) + m_r \Sigma_S(k^2) + \delta Z_\psi(\not{k} - m_r) - m_r \delta Z_m \tag{10.81}$$

  In this scheme the mass renormalization constant is determined from the condition

$$\Sigma_r(k^2) u(k)|_{k^2 = m_r^2} = 0 \tag{10.82}$$

  i.e.

$$\delta Z_m = \Sigma_V(m_r^2) + \Sigma_S(m_r^2) \tag{10.83}$$

  The field renormalization constant can be determined from the condition that the propagator has residue one which implies

$$\frac{1}{\not{k} - m_r} \Sigma_r(k^2) u(k)|_{k^2 = m_r^2} = 0 \tag{10.84}$$

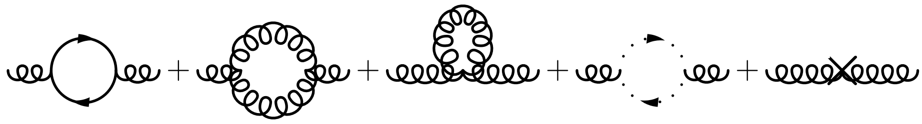
  The on-shell scheme is often used in QED and the electroweak sector of the Standard Model. In QCD it may be used for the top quark, but it is not useful for the light quarks, which cannot be observed freely.

The usual choice in QCD is the $\overline{\text{(MS)}}$ scheme. In this scheme the renormalized quark self energy is

$$\Sigma_r(k^2) = -\frac{\alpha_s}{4\pi} C_F \not{k} \left(\log\left(\frac{-k^2}{\mu^2}\right) - 1\right) \tag{10.85}$$

Gluon field renormalization

For the gluon field renormalization, a fermionic one-loop diagram, two gluon diagrams, a ghost diagram and a counterterm diagram need to be calculated:

$\Sigma_{A,r}(k^2) =$  (10.86)

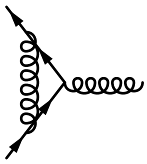
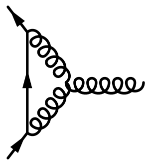
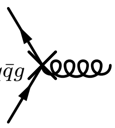
We here just quote the result of the counterterm in the $\overline{\text{(MS)}}$ scheme

$$\delta Z_A^{\overline{\text{(MS)}}} = -\frac{\alpha_s}{4\pi}\left(\frac{4}{3}T_F N_f - \frac{5}{3}C_A\right)\Delta_\varepsilon \tag{10.87}$$

where N_f is the number of quark flavours.

Coupling constant renormalization

The renormalization constant for the strong coupling constant, δZ_{g_s}, can be obtained from the NLO corrections to the quark-gluon vertex

$\Lambda_{qqg,r} =$  $+$  $+\delta Z_{q\bar{q}g}$  (10.88)

with

$$\delta Z_{q\bar{q}g} = \delta Z_\psi + \delta Z_{g_s} + \frac{1}{2}\delta Z_A \tag{10.89}$$

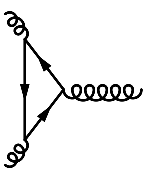
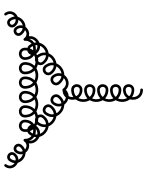
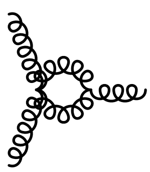
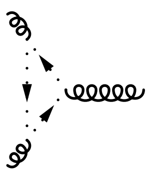
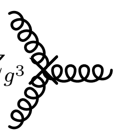
As mentioned above, the vertex integrals contain both ultraviolet singularities from large loop momenta, but also so-called infrared singularities. These arise for massless particles in the loop whose momentum can become very small (so-called soft singularities) or collinear to a massless external particle. Extracting the UV divergence of the one-loop vertex function (e.g. by introducing quark masses to regularize collinear singularities and setting external momenta to zero) one finds in the $\overline{\text{MS}}$ scheme

$$\delta Z_{q\bar{q}g}^{\overline{\text{MS}}} = -\frac{\alpha_s}{4\pi}(C_A + C_F)\Delta_\varepsilon \tag{10.90}$$

Using the results for the field renormalization constants δZ_ψ and δZ_A allows to obtain the coupling constant renormalization:

$$\delta Z_{g_s}^{\overline{\text{MS}}} = \frac{\alpha_s}{4\pi}\left(\frac{2}{3}T_F N_f - \frac{11}{6}C_A\right)\Delta_\varepsilon \tag{10.91}$$

There are several diagrams contributing to the three-gluon vertex:

 $+$  $+$  $+ \cdots +$  $+\ \delta Z_{g^3}$ 

(10.92)

Our previous results allow to calculate the counterterm for this vertex in terms of the coupling-constant renormalization and the gluon field renormalization:

$$\delta Z_{g^3} = \delta Z_{g_s} + \frac{3}{2}\delta Z_A = -\left(\frac{4}{3}T_F N_f - \frac{2}{3}C_A\right)\Delta_\varepsilon \tag{10.93}$$

It is a non-trivial result of gauge invariance and the renormalizability of QCD that the counterterm determined from the previously calculated renormalization constants cancels the UV divergences of the one-loop diagrams contributing to the vertex. The same holds true for the four-gluon vertex.

10.4 NLO corrections to $e^+e^- \to$ Hadrons

We can now return to the calculation of the NLO corrections to $e^+e^- \to$ Hadrons which consist of the the virtual corrections to $e^+e^- \to q\bar{q}$ (10.4) and the real-correction process $e^+e^- \to q\bar{q}g$ (10.5). Both pieces contain infrared singularities that cancel when both contributions are added up.

10.4.1 Virtual corrections

The vertex corrections have been calculated in (10.76) and can be written as

$$\Delta_V \mathcal{M} = \mathcal{M}_0 \delta_V \tag{10.94}$$

with the virtual correction factor

$$\delta_V = -\frac{\alpha_s}{4\pi} C_F (4\pi)^\varepsilon \Gamma(1+\varepsilon)\left(-\frac{Q^2 + \mathrm{i}\epsilon}{\mu^2}\right)^{-\varepsilon}\left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8 - \frac{\pi^2}{3}\right] \tag{10.95}$$

where $\mathcal{M}_0$ is the LO matrix element and $Q = p_1 + p_2$.

There are no corrections from the self-energy diagrams and the quark-field renormalization. This can be seen most easily by working with bare quark fields (where no quark-field renormalization is performed) and using that the self-energies for external on-shell particles vanish. Alternatively, one performs the field renormalization which results in a vertex counterterm $\delta Z_{qq\gamma} = \delta Z_\psi$. Using the LSZ formalism (see e.g. [1]), the external legs are amputated and replaced by the square root of the residue of the propagator. For massless quarks the unrenormalized self-energy vanishes, so the square root of the residue of the renormalized propagator is given by $1 - \frac{1}{2}\delta Z_\psi$. This precisely cancels the vertex counterterm.

The virtual corrections to the cross section are then given by

$$\Delta_R \sigma(e^-e^+ \to \text{hadrons}) = \sigma_0^{(\varepsilon)} 2\mathrm{Re}\delta_V \tag{10.96}$$

Since the real corrections below are calculated in d dimensions to regularize infrared divergences, here for consistency the leading order cross section in d dimensions, $\sigma_0^{(\varepsilon)}$ appears, but the explicit expression is not needed.

The real part of the virtual corrections can be taken using the relation

$$\frac{1}{\varepsilon^2}\left(-\frac{Q^2+\mathrm{i}\epsilon}{\mu^2}\right)^{-\varepsilon} = \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\left(\log\frac{Q^2}{\mu^2}+\mathrm{i}\pi\right) + \left(\log\frac{Q^2}{\mu^2}+\mathrm{i}\pi\right)^2 \tag{10.97}$$

Therefore

$$\mathrm{Re}\frac{1}{\varepsilon^2}\left(-\frac{Q^2+\mathrm{i}\epsilon}{\mu^2}\right)^{-\varepsilon} = \frac{1}{\varepsilon}^2\left(\frac{Q^2}{\mu^2}\right)^{-\varepsilon} - \frac{\pi^2}{2} \tag{10.98}$$

We therefore get the result for the virtual correction factor

$$2\mathrm{Re}\delta_V = -\frac{\alpha_s}{2\pi}C_F\Gamma(1+\varepsilon)\left(\frac{Q^2}{\mu^2}\right)^{-\varepsilon}\left[\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+4-\frac{4\pi^2}{3}\right] \tag{10.99}$$

10.4.2 Real corrections and IR singularities

We have computed the helicity amplitudes for $e^-e^+ \to q\bar{q}$ and $e^-e^+ \to q\bar{q}g$ for some helicity combinations in Section 6.3. The four point amplitude is

$$\begin{aligned}\mathrm{i}\mathcal{M}(e^{-,L}_{p_1}e^{+,R}_{p_2} \to q^R_{k_1}\bar{q}^L_{k_2}) &= 2\mathrm{i}e^2\frac{\langle p_2k_2\rangle^2}{\langle p_1p_2\rangle\langle k_2k_1\rangle}\\ &= 2\mathrm{i}e^2\frac{[p_1k_1]^2}{[p_1p_2]\,[k_1k_2]}\end{aligned} \tag{10.100}$$

The real-emission amplitudes for the two different gluon polarizations are

$$\mathrm{i}\mathcal{M}(e^{-,L}_{p_1}e^{+,R}_{p_2} \to q^R_{k_1}\bar{q}^L_{k_2}g^+_{k_3}) = \mathrm{i}e^2Q_qg_sT^{a,i_1}_{i_2}\sqrt{2}\frac{2\langle p_2k_2\rangle^2}{\langle p_1p_2\rangle\langle k_2k_3\rangle\langle k_3k_1\rangle} \tag{10.101}$$

$$\mathrm{i}\mathcal{M}(e^{-,L}_{p_1}e^{+,R}_{p_2} \to q^R_{k_1}\bar{q}^L_{k_2}g^-_{k_3}) = \mathrm{i}e^2Q_qg_sT^{a,i_1}_{i_2}\sqrt{2}\frac{2\,[p_1k_1]^2}{[p_1p_2]\,[k_2k_3]\,[k_3k_1]} \tag{10.102}$$

The amplitudes for different fermion polarizations are obtained by exchanging the spinors in the numerator.

The colour-summed square of the real-emission helicity amplitudes, summed/averaged over helicities, gives

$$\begin{aligned}&\frac{1}{4}\sum_{\text{colors, hel.}}|\mathcal{M}(e^-_{p_1}e^+_{p_2} \to q_{k_1}\bar{q}_{k_2}g_{k_3})|^2\\ &= \frac{(4N_cQ_q^2e^4)}{(p_1+p_2)^2}(2g_s^2C_F)\frac{(k_2\cdot p_2)^2+(k_1\cdot p_1)^2+(k_2\cdot p_1)^2+(k_1\cdot p_2)^2}{(k_2\cdot k_3)(k_1\cdot k_3)}\end{aligned} \tag{10.103}$$

The real-emission cross section can be written in terms of the energy fractions

$$x_i = \frac{2k_i\cdot Q}{Q^2} \tag{10.104}$$

with $Q = p_1 + p_2 = k_1 + k_2 + k_3 = k_{1,3}$. The energy fractions satisfy

$$x_1 + x_2 + x_3 = 2. \tag{10.105}$$

Using identities such as

$$(1 - x_1) = \frac{Q^2 - 2k_1 \cdot Q}{Q^2} = \frac{k_{1,3}^2 - 2k_1 \cdot (k_2 + k_3)}{Q^2} = \frac{2k_2 \cdot k_3}{Q^2} \tag{10.106}$$

one finds that the matrix element squared is proportional to

$$|\mathcal{M}|^2 \sim \frac{1}{(1 - x_1)(1 - x_2)} \tag{10.107}$$

One sees that the matrix element becomes singular in the limits

$$\begin{aligned} \text{collinear} :& (k_1 \cdot k_3) \to 0, \quad x_2 \to 1 \\ & (k_2 \cdot k_3) \to 0, \quad x_1 \to 1 \\ \text{soft} :& k_3 \to 0, \qquad\quad\; x_3 \to 0 \end{aligned} \tag{10.108}$$

For consistency with the treatment of the virtual corrections, dimensional regularization is also used for the real corrections, i.e. the phase-space volume is modified to

$$\mathrm{d}\Phi_f = \left(\prod_{l=1}^{n} \frac{\mathrm{d}^{d-1} k_l}{(2\pi)^{d-1}(2k_l^0)} \right) (2\pi)^d \mu^{n(4-d)} \delta(p_1 + p_2 - \sum k_f) \tag{10.109}$$

Performing also the computation of the matrix element in d-dimensions one can show that the d-dimensional phase-space integral over the phase-space of the gluons gives the real corrections to the cross section as [7]

$$\Delta_R \sigma(e^- e^+ \to \text{hadrons}) = \sigma_0^{(\varepsilon)} \delta_R \tag{10.110}$$

with the real-correction factor

$$\begin{aligned} \delta_R &= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2} \right)^{\varepsilon} \frac{1}{\Gamma(1-\varepsilon)} \int_0^1 \mathrm{d}x_1 \int_{1-x_1}^1 \mathrm{d}x_2 \frac{1}{((1-x_1)(1-x_2)(1-x_3))^{\varepsilon}} \frac{x_1^2 + x_2^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)} \\ &= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2} \right)^{\varepsilon} \Gamma(1+\varepsilon) \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \frac{4\pi^2}{3} \right] \end{aligned} \tag{10.111}$$

10.4.3 NLO corrections to the R-ratio

At leading order the R-ration is given by

$$R_{\mathrm{LO}} = \frac{\sigma_0(e^+e^- \to \text{Hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_q N_c Q_q^2 \tag{10.112}$$

where the sum is over all quarks with $m_q < E_{CM}/2$.

Adding the real and virtual NLO corrections all $1/\varepsilon$ poles drop out. After expanding the NLO corrections in ε the limit $d \to 4$ can be taken in the LO cross section. The result is

$$\Delta_{\mathrm{NLO}}(e^+e^- \to \mathrm{Hadrons}) = \sigma_0^{(\varepsilon)}(\delta_R + 2\mathrm{Re}\delta_V) = \sigma_0 \frac{3\alpha_s C_F}{4\pi} = \sigma_0 \frac{\alpha_s}{\pi} \tag{10.113}$$

A comparison of the measured R-ratio to the QCD prediction therefore allows to measure the value of the strong coupling constant. There is, however, one more complication that appears only if higher orders in QCD are considered and will be discussed in the next section.

10.5 Renormalization group

In the NLO calculation of the R-ratio the logarithms of the form $\log\left(\frac{Q^2}{\mu^2}\right)$ dropped out completely in the sum of real and virtual corrections. This is an artifact which appears because we calculated the QCD corrections to a QED observable, where the renormalization of the QCD coupling did not play a role. In general, since the R-ratio is a dimensionless observable one expects a series expansion in α_s of the form

$$R = R_0 \left(1 + \alpha_s r_1(t) + \alpha_s^2 r_2(t) + \dots\right) \tag{10.114}$$

with the variable $t = \frac{Q^2}{\mu^2}$. Since the scale μ is arbitrary, the observable R must not depend on it:

$$\mu^2 \frac{\mathrm{d}R}{\mathrm{d}\mu^2} = 0. \tag{10.115}$$

The functional dependence of functions r_n is constrained by this so-called **renormalization-group equation**, which we will briefly discuss here following [6].

10.5.1 Running coupling

To evaluate the condition (10.115), one must take into account that the relation of the renormalized dimensionless coupling α_s to the bare coupling $\alpha_{s,0}$ involves the scale μ, according to (10.27):

$$\alpha_s = \mu^{-2\varepsilon}\alpha_{s,r} = \mu^{-2\varepsilon} Z_g^{-2} \alpha_{s,0}, \tag{10.116}$$

Since $\alpha_{s,0}$ is the bare, unrenormalized, coupling defined without reference to μ, we have

$$\frac{\mathrm{d}\alpha_{s,0}}{\mathrm{d}\mu} = 0 \tag{10.117}$$

This implies that the dimensionless coupling satisfies the differential equation

$$\mu^2 \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu^2} = -\varepsilon\alpha_s - 2\frac{\mu^2}{Z_g}\frac{\mathrm{d}Z_g}{\mathrm{d}\mu^2}\alpha_s \equiv \beta, \tag{10.118}$$

which is called the **renormalization-group equation** for α_s. This equation defines the so-called **beta function** $\beta(\alpha_s)$.

In the $\overline{\text{MS}}$ -scheme the renormalization constant Z_g depends on the scale μ only through α_s so that

$$\frac{\mu^2}{Z_g}\frac{\mathrm{d}Z_g}{\mathrm{d}\mu^2} = \frac{1}{Z_g}\frac{\partial Z_g}{\partial\alpha_s}\mu^2\frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu^2} = \frac{1}{Z_g}\frac{\partial Z_g}{\partial\alpha_s}\beta. \tag{10.119}$$

At NLO in QCD, the coupling constant renormalization constant in the $\overline{\text{MS}}$ -scheme is given by (10.91)

$$Z_g = 1 + \frac{\alpha_s}{4\pi}\left(\frac{2}{3}T_F N_f - \frac{11}{6}C_A\right)\Delta_\varepsilon \tag{10.120}$$

so that

$$\frac{\partial Z_g}{\partial\alpha_s} = \frac{1}{(4\pi)}\left(\frac{2}{3}T_F N_f - \frac{11}{6}C_A\right)\Delta_\varepsilon \tag{10.121}$$

The β-function can be computed perturbatively in g_s:

$$\begin{aligned}\beta = -\varepsilon\alpha_s - 2\alpha_s\frac{1}{Z_g}\frac{\partial Z_g}{\partial\alpha_s}\beta &= -\varepsilon\alpha_s - \frac{2\alpha_s}{(4\pi)}\left(\frac{2}{3}T_F N_f - \frac{11}{6}C_A\right)\Delta_\varepsilon \underbrace{\beta}_{-\varepsilon\alpha_s+\ldots} + \ldots \\ &= -\frac{\alpha_s^2}{4\pi}\left(\frac{11}{3}C_A - \frac{4}{3}T_F N_f\right) + \ldots\end{aligned} \tag{10.122}$$

One sees that the β function is finite for $\varepsilon \to 0$ and independent of μ in the $\overline{\text{MS}}$-scheme. Defining the perturbative expansion of the beta function as

$$\beta(\alpha_s) = -\alpha_s\sum_{n=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n\beta_n \tag{10.123}$$

we see that

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f. \tag{10.124}$$

The coefficients β_n have been calculated up to $n = 3$ [14].

The leading-order equation for the **running coupling** $\alpha_s(\mu)$ can be solved in terms of the value at a reference scale $\alpha_s(\mu_0)$ by a separation of variables:

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu^2} = -\alpha_s^2\frac{\beta_0}{4\pi} \tag{10.125}$$

$$\Rightarrow \quad \frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\mu_0)} = \frac{\beta_0}{4\pi}\log\left(\frac{\mu^2}{\mu_0^2}\right) \tag{10.126}$$

or

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\alpha_s(\mu_0)}{4\pi}\beta_0\log\left(\frac{\mu^2}{\mu_0^2}\right)} \tag{10.127}$$

For the case of QCD we have $C_A = N_c = 3$, $T_F = \frac{1}{2}$ so that

$$\beta_0 = 11 - \frac{2}{3} n_f > 0 \tag{10.128}$$

which is positive since the number of quark flavours is less than 16. The solution (10.127) implies that given a finite value of $\alpha_s(\mu_0)$ at some reference scale μ_0, the running coupling diverges at some scale $\Lambda < \mu_0$ where

$$\Lambda = \exp\left(\frac{2\pi}{\beta_0 \alpha_s(\mu_0)}\right) \tag{10.129}$$

The LO-solution to the running coupling is therefore also often written as

$$\alpha_s(\mu) = \frac{1}{\frac{\beta_0}{4\pi}\log\left(\frac{\mu^2}{\Lambda^2}\right)} \tag{10.130}$$

The fact that the running coupling becomes smaller for larger scales μ in QCD is the famous property called **asymptotic freedom** whose discovery led to the Nobel Prize for Gross, Politzer and Wilzcek. However, at this point in our discussion the physical interpretation of the dependence of the coupling on the supposedly arbitrary scale μ is still quite obscure.

10.5.2 Resummation-group for the R-ratio

The independence of the R-ratio on μ (10.115) implies

$$0 = \mu^2 \frac{\mathrm{d}R}{\mathrm{d}\mu^2} = \left(\mu^2 \frac{\partial R}{\partial \mu^2} + \beta \frac{\partial R}{\partial \alpha_s}\right) \tag{10.131}$$

since R depends on μ both explicitly through the ratio $t = \frac{Q^2}{\mu^2}$ and through α_s. Inserting the series (10.114) and taking into account that $\beta = \mathcal{O}(\alpha_s^2)$ this equation implies the differential equations for the functions $r_i(t)$:

$$\begin{aligned} \mu^2 \frac{\partial r_1}{\partial \mu^2} &= 0 \quad \Rightarrow r_1 = \text{const.} \equiv a_1 \\ \mu^2 \frac{\partial r_2}{\partial \mu^2} + \frac{\beta_0}{4\pi} r_1 &= 0 \quad \Rightarrow r_2 = -\frac{\beta_0}{4\pi} a_1 \log t + a_2 \\ &\dots \end{aligned} \tag{10.132}$$

with an integration constant a_2. Therefore the renormalization group analysis explains our previous result that $r_1 = \frac{1}{\pi}$ is a constant and allows to predict that the expansion up to NNLO has the form

$$R(s) = R_0 \left\{1 + \frac{\alpha_s(\mu)}{\pi} + \alpha_s^2(\mu)\left[-\frac{\beta_0}{4\pi^2}\log\left(\frac{s}{\mu^2}\right) + a_2\right] + \dots\right\} \tag{10.133}$$

The coefficient of the logarithmic term at NNLO is therefore predicted by the NLO calculation, while the determination of the constant a_2 requires an actual NNLO calculation.

For the choice $\mu^2 = s$ the NNLO prediction simplifies to

$$R(s) = R_0 \left\{ 1 + \frac{\alpha_s(\sqrt{s})}{\pi} + \alpha_s^2(\sqrt{s})a_2 + \dots \right\} \tag{10.134}$$

Using this relation, one can determine $\alpha_s(\sqrt{s})$ from the measurement of the R-ratio. Note, however, that the definition of α_s depends on the renormalization scheme in higher orders. The current world average for $\sqrt{s} = M_Z$ (including also measurements from other processes) is [14]

$$\alpha_s(M_Z) = 0.1185 \pm 0.0006. \tag{10.135}$$

The fact that $\alpha_s(M_Z) < 1$ justifies a posteori the perturbative calculation of the R-ratio.

10.5.3 Resummation-group improvement

We can now turn to the physical interpretation of the running coupling constant. The result (10.133) allows in principle to predict $R(s)$ at arbitrary energies, given $\alpha_s(M_Z)$ as input. However, for energies very different from M_Z, the logarithm can become very large so that

$$\frac{\alpha_s(M_Z)}{4\pi}\beta_0 \log\left(\frac{s}{M_Z^2}\right) \gg 1 \tag{10.136}$$

Therefore the expansion in $\alpha_s(M_Z)$ is not a reliable perturbative expansion for $s \gg M_Z$ or $s \ll M_Z$ since corrections from subsequent orders not yet computed can be just as large as the computed ones. However, setting $\mu^2 = s$ and using the formula for the running coupling (10.127) (and its higher-order generalization) one sees that in this way the dangerous logarithms (10.136) are summed up to arbitrary order in the coupling $\alpha_s(M_Z)$. Therefore a more reliable prediction for the R-ratio at an arbitrary energy is given by the expression in terms of the running coupling at the appropriate scale (10.134)[1]. In this way the knowledge of the dependence of the running coupling on the artificial scale μ can be used to improve the prediction for physical processes, and it is useful to introduce the notion of an energy-dependent running coupling constant.

Because of the positive sign of β_0 the running coupling $\alpha_S(\sqrt{s})$ becomes larger for increasing energy and grows for smaller energies. This is confirmed by experimental measurements over several orders of magnitude of the energy scale, see Figure (10.1). At some small scale the running coupling grows large so the perturbative expansion breaks down for small energies. This behaviour at small scales is qualitatively in agreement with the confinement of quarks and gluons in hadrons at small energies.

[1]Other choices of the scale have been proposed to further improve the behaviour of the perturbative expansion, see e.g section 3.4.6 of [7].

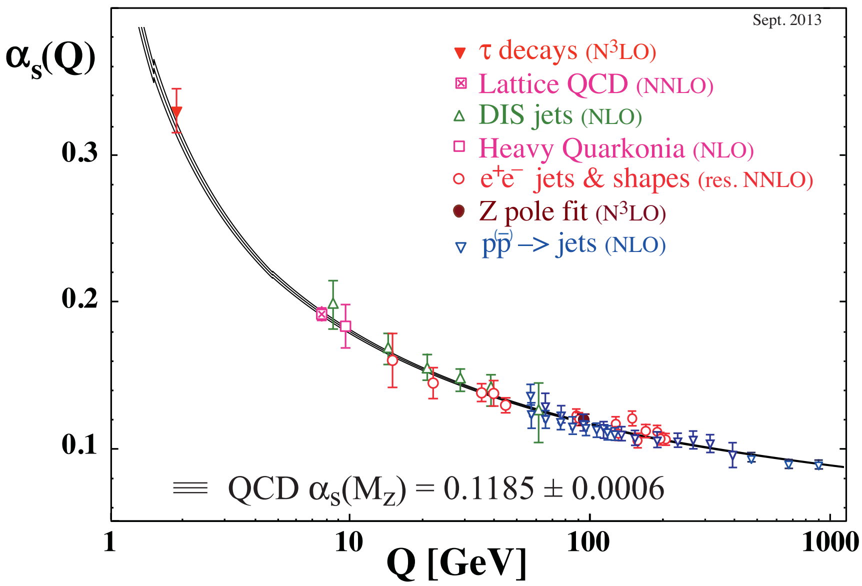


Figure 10.1: Measurements of the strong couplings at different scales, taken from [14].

Appendix A

BRST symmetry and Slavnov-Taylor identities

A.1 Gauge fixing and ghost Lagrangian

Recall that the quantization of non-abelian gauge theories requires the introduction of a gauge fixing term and Fadeev-Popov ghost fields. For a covariant gauge fixing, the gauge fixing Lagrangian and the Fadeev-Popov Lagrangian are given by (4.73) and (4.74), resepctively:

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi}(\partial_\mu A^{a,\mu})^2, \tag{A.1}$$

$$\mathcal{L}_{\mathrm{FP}} = (\partial^\mu \bar{c}^a) D^{(\mathrm{ad})}_{ab,\mu} c^b = (\partial^\mu \bar{c}^a)(\partial_\mu \delta_{ab} + g_s f^{abc} A_{c,\mu}) c^b \tag{A.2}$$

The ghost fields are anticommuting scalars. This implies that

$$(\bar{c}^a c^b)^\dagger = c^{b\dagger} \bar{c}^{a\dagger} = -\bar{c}^{a\dagger} c^{b\dagger} \tag{A.3}$$

The Lagrangian is hermitian for the assignment

$$c^{a\dagger} = c^a \qquad\qquad \bar{c}^{a\dagger} = -\bar{c}^a \tag{A.4}$$

It is not consistent with a hermitian interaction term with the gauge boson to take the antighost as the conjugate of the ghost. The antighost could be made hermitian by a redefinition $\bar{c} \to \mathrm{i}\bar{c}$ but we will keep the form of the Lagrangian given above.

The gauge-fixing Lagrangian can be rewritten in terms of the so-called **Nakanishi-Lautrup auxiliary fields** B_a as

$$\mathcal{L}_{\mathrm{gf}} = B^a f^a + \frac{\xi}{2}(B^a)^2 \tag{A.5}$$

as can be seen using the equation of motion for the auxiliary fields

$$0 = \frac{\partial \mathcal{L}}{\partial B^a} = \xi B^a + f^a \tag{A.6}$$

A.2 BRST transformations

A.2.1 Definition of the BRST transformation

The gauge-fixed QCD Lagrangian including the Fadeev-Popov Lagrangian is invariant under a global transformation parametrized by a Grassmann-valued parameter θ. This transformation was discovered by Becchi, Rouet, Stora, and Tyutin (BRST). We write the transformations of a general field Ψ as

$$\Delta_\theta \Psi = \theta \delta_{\mathrm{B}} \Psi \tag{A.7}$$

The transformations of the physical fields are obtained from the infinitesimal gauge transformations by replacing the parameters ω^a by the product of the ghost fields and the Grassmann number θ:

$$\delta_{\mathrm{B}} A^a_\mu = \partial_\mu c^a + g_s f^{abc} c^b A^c_\mu \tag{A.8}$$
$$\delta_{\mathrm{B}} Q(x) = -\mathrm{i} g_s c^a(x) T^a Q(x) \tag{A.9}$$

The transformations of the ghost fields and the auxiliary fields are taken as

$$\delta_{\mathrm{B}} c^a = \frac{1}{2} g_s f^{abc} c^b c^c \tag{A.10}$$
$$\delta_{\mathrm{B}} \bar{c}^a = B^a \tag{A.11}$$
$$\delta_{\mathrm{B}} B^a = 0 \tag{A.12}$$

A.2.2 BRST charge

We introduce the generator of BRST transformations Q_{B}, the so-called BRST charge:

$$\Delta_\theta \Psi = \theta \delta_{\mathrm{B}} \Psi \equiv [\mathrm{i}\theta Q_{\mathrm{B}}, \Psi]. \tag{A.13}$$

The BRST charge can be constructed explicitly using the Noether theorem [21]. The BRST transformations of bosonic fields are generated by commutators with the BRST charge, the transformations of fermionic fields by anticommutators:

$$[Q_{\mathrm{B}}, \Phi]_\pm = -\mathrm{i}\delta_{\mathrm{B}} \Phi \tag{A.14}$$

With the above definition of the conjugations of the ghost and anti-ghost fields, the BRST charge is hermitian:

$$[Q_{\mathrm{B}}, A^a]^\dagger = \mathrm{i}(\partial_\mu c^a + g_s f^{abc} c^b A^c_\mu) = [A^a, Q_{\mathrm{B}}] \tag{A.15}$$
$$\{Q_{\mathrm{B}}, Q\}^\dagger = -g_s Q^\dagger T^a c^a = g_s c^a Q^\dagger T^a = \{Q^\dagger, Q_{\mathrm{B}}\} \tag{A.16}$$
$$\{Q_{\mathrm{B}}, \bar{c}^a\}^\dagger = \mathrm{i} B^a = \{\bar{c}^{a\dagger}, Q_{\mathrm{B}}\} \tag{A.17}$$
$$\{Q_{\mathrm{B}}, c^a\}^\dagger = \frac{\mathrm{i}}{2} g_s f^{abc} c^c c^b = -\frac{\mathrm{i}}{2} g_s f^{abc} c^b c^c = \{c^{a\dagger}, Q_{\mathrm{B}}\} \tag{A.18}$$

The BRST transformation of products of fields is defined as

$$\begin{aligned}\Delta_\theta(\Psi_1 \dots \Psi_n) &= [\mathrm{i}\theta Q_{\mathrm{B}}, \Psi_1 \dots \Psi_n] \\ &= \theta(\delta_{\mathrm{B}}\Psi_1) \dots \Psi_n + \dots \theta(-1)^{s_i}\Psi_1 \dots (\delta_{\mathrm{B}}\Psi_i) \dots \Psi_n \\ &\equiv \theta\delta_{\mathrm{B}}(\Psi_1 \dots \Psi_n)\end{aligned} \tag{A.19}$$

where s_i counts the number of fermionic fields before Ψ_i. The last line defines the action of δ_{B} on products of fields.

Note that we have

$$\begin{aligned}\delta_{\mathrm{B}}^2(\Psi_1\Psi_2) &= \delta_{\mathrm{B}}\left[(\delta_{\mathrm{B}}\Psi_1)\Psi_2 + (-1)^{s_1}\Psi_1(\delta_{\mathrm{B}}\Psi_2)\right] \\ &= (\delta_{\mathrm{B}}^2\Psi_1)\Psi_2 - (-1)^{s_1}(\delta_{\mathrm{B}}\Psi_1)(\delta_{\mathrm{B}}\Psi_2) + (-1)^{s_1}(\delta_{\mathrm{B}}\Psi_1(\delta_{\mathrm{B}}\Psi_2) + \Psi_1(\delta_{\mathrm{B}}^2\Psi_2) \\ &= (\delta_{\mathrm{B}}^2\Psi_1)\Psi_2 + \Psi_1(\delta_{\mathrm{B}}^2\Psi_2)\end{aligned} \tag{A.20}$$

A.2.3 Properties

The BRST transformation has the following properties:

1. it leaves the Lagrangian invariant
2. it is **nilpotent**, i.e. for any field one has
$$\delta_{\mathrm{B}}^2\Phi = 0 \tag{A.21}$$
Because of (A.20) this implies automatically that $\delta_{\mathrm{B}}^2 F = 0$ for any functional F of the fields.
3. The sum of the gauge-fixing and ghost Lagrangians can be written as
$$\mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{FP}} = \delta_{\mathrm{B}}\mathcal{F} \tag{A.22}$$
with
$$\mathcal{F} = \bar{c}^a(f^a + \frac{\xi}{2}B^a) \tag{A.23}$$
For the choice of the gauge fixing function $f^a = \partial_\mu A^{a,\mu}$ this is easily seen:
$$\delta_{\mathrm{B}}\mathcal{F} = B^a(\partial_\mu A^{a,\mu} + \frac{\xi}{2}B^a) - \bar{c}^a\partial_\mu(\partial^\mu c^a + g_s f^{abc}c^b A^{c,\mu}) \tag{A.24}$$
For a general gauge fixing functional one uses
$$\delta_{\mathrm{B}} f^a[A] = \int \mathrm{d}^4 y \frac{\delta f^a[\mathcal{A}'_\mu(x)]}{\delta\omega^b(y)}\theta c^b = \mathcal{M}^{ab}\theta c^b \tag{A.25}$$
so that
$$\delta_{\mathrm{B}}\mathcal{F} = B^a(f^a + \frac{\xi}{2}B^a) - \bar{c}^a\mathcal{M}^{ab}c^b \tag{A.26}$$

The BRST invariance of the Lagrangian follows from the two other properties and the fact that the classical QCD Lagrangian is invariant by construction.

Proof of nilpotency

It remains to be shown that the BRST transformation is nilpotent.

The nilpotency is obvious for the antighost and the auxiliary field (in fact, this is the motivation for introducing the auxiliary field, since the nilpotency holds only after use of the equations of motion if the formulation without auxiliary fields is used).

For the quark field we have

$$\begin{aligned}\delta_{\mathrm{B}}^2 Q &= -\mathrm{i}g_s\left((\delta_{\mathrm{B}}c^a)T^aQ - c^aT^a\delta_{\mathrm{B}}Q\right)\\ &= -\mathrm{i}g_s^2\Big(\frac{1}{2}c^bc^cf^{abc}T^aQ + \mathrm{i}\,\underbrace{c^bc^cT^bT^c}_{\frac{1}{2}c^bc^c[T^b,T^c]}\,Q\Big)\\ &= 0\end{aligned} \tag{A.27}$$

For the gluon field, the repeated application of the BRS transformation gives

$$\begin{aligned}\delta_{\mathrm{B}}^2 A_\mu^a &= \partial_\mu\delta_{\mathrm{B}}c^a + g_sf^{abc}(\delta_{\mathrm{B}}c^b)A_\mu^c - g_sf^{abc}c^b\delta_{\mathrm{B}}A_\mu^c\\ &= \frac{g_s}{2}\left[f^{abc}\partial_\mu(c^bc^c) + g_sf^{abc}f^{bde}c^dc^eA_\mu^c - 2f^{abc}c^b\left(\partial_\mu c^c + g_sf^{cde}c^dA_\mu^e\right)\right]\end{aligned} \tag{A.28}$$

This vanishes as can be seen separately for the derivative terms and the terms with the gauge fields, using the anticommuting nature of the ghosts

$$f^{abc}\left(\partial_\mu(c^bc^c) - 2c^b\partial_\mu c^c\right) = f^{abc}\left(\partial_\mu(c^bc^c) - c^b(\partial_\mu c^c) - (\partial_\mu c^b)c^c\right) \quad = 0 \tag{A.29}$$

$$\begin{aligned}f^{abc}f^{bde}c^dc^eA_\mu^c - 2f^{abc}f^{cde}c^bc^dA_\mu^e &= A_\mu^c\left(f^{abc}f^{bde}c^dc^e + 2f^{abe}f^{edc}c^dc^b\right)\\ &= A_\mu^cc^dc^e\left(f^{abc}f^{bde} + 2f^{aeb}f^{bdc}\right)\\ &= A_\mu^cc^dc^e\left(f^{abc}f^{bde} + f^{aeb}f^{bdc} - f^{adb}f^{bec}\right) \quad = 0\end{aligned} \tag{A.30}$$

In the last line the Jacobi identity

$$f^{bcd}f^{ade} + f^{abd}f^{cde} + f^{cad}f^{bde} = 0. \tag{A.31}$$

was used. Since the term $\sim cA$ in the transformation law of the gauge field is the same as for a matter field in the adjoint representation, the cancellations in this case work in the same way as for the quark term, up to replacing the generators in the fundamental by those in the adjoint representation.

The last step is the proof of nilpotency for the transformation of the ghost fields:

$$\begin{aligned}\delta_{\mathrm{B}}^2 c^a &= \frac{1}{2}g_sf^{abc}\left[(\delta_{\mathrm{B}}c^b)c^c - c^b(\delta_{\mathrm{B}}c^c)\right]\\ &= \frac{1}{4}g_s^2f^{abc}\left[f^{bde}c^dc^ec^c - f^{cde}c^bc^dc^e\right]\\ &= \frac{1}{2}g_s^2f^{abc}f^{bde}c^cc^dc^e = 0\end{aligned} \tag{A.32}$$

Since the product of three ghost fields does not change sign under cyclic permutations, this expression vanishes as a result of the Jacobi identity.

A.3 BRST symmetry and states in a gauge theory

The vector space of states of a gauge theory contains four modes of the gauge field and the ghosts and antighosts, whereas the classification of states using the representations of the Poincaré group shows that only two transverse polarizations of the vector fields should appear. The BRST symmetry allows to define "physical" states consistently and allows to show that the unphysical states decouple.

A.3.1 Physical states

The nilpotency of the BRST transformation implies also

$$Q_{\mathrm{B}}^2 = 0 \tag{A.33}$$

This can be seen by computing the double commutator, for instance for a bosonic field,

$$0 = \delta_{\mathrm{B}}^2 \Phi = \{\mathrm{i}Q_{\mathrm{B}}, [\mathrm{i}Q_{\mathrm{B}}, \Phi]\} = -(Q_{\mathrm{B}}^2\Phi - Q_{\mathrm{B}}\Phi Q_{\mathrm{B}} + Q_{\mathrm{B}}\Phi Q_{\mathrm{B}} + \Phi Q_{\mathrm{B}}^2) = -[Q_{\mathrm{B}}^2, \Phi] \tag{A.34}$$

This implies $Q_{\mathrm{B}}^2 = 0$. Note that the BRST transformation changes the ghost number by one, so that Q_{B}^2 must have ghost number two. This excludes the possibility that $Q_{\mathrm{B}}^2 \propto \mathbf{1}$.

Because of the nilpotency of Q, states that are obtained by applying Q_{B} to another arbitrary state (so called 'BRS exact states') have vanishing norm:

$$|\psi\rangle = Q_{\mathrm{B}}\,|\eta\rangle : \quad \langle\psi|\psi\rangle = 0 \qquad \forall\,|\eta\rangle \tag{A.35}$$

States that are annihilated by the BRS charge are called 'BRS closed'. They are orthogonal to the exact states:

$$\langle\psi|\phi\rangle = \langle\eta|Q_{\mathrm{B}}|\phi\rangle = 0 \qquad \forall\,|\psi\rangle = Q_{\mathrm{B}}\,|\eta\rangle \quad , \quad Q_{\mathrm{B}}\,|\phi\rangle = 0 \tag{A.36}$$

Therefore we can decompose the Hilbert space into orthogonal subspaces. Because of the nilpotency of Q_{B}, a closed state stays closed if one adds an arbitrary exact state.

One can show (see e.g. [21]) that provided the BRS closed states have positive norm, it is consistent to define the **physical** states of the theory as closed states modulo exact states:

$$\begin{aligned} Q_{\mathrm{B}}\,|\psi_{\mathrm{phys}}\rangle &= 0 \\ |\psi_{\mathrm{phys}}\rangle &\sim |\psi_{\mathrm{phys}}\rangle + Q_{\mathrm{B}}\,|\eta\rangle \end{aligned} \tag{A.37}$$

In mathematical terms, this is the **cohomology** of the operator Q.

A.3.2 Asymptotic fields

Consider the asymptotic in/out states that satisfy the free equations of motion.

They admit the same mode decomposition as the free fields, i.e. for the gauge field (suppressing colour indices)

$$A^\mu(x) = \sum_{\lambda=\pm,,L,S} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0} \left(a_\lambda(\vec{p})\epsilon_\lambda^\mu(p) e^{-ipx} + a_\lambda^\dagger(\vec{p})\epsilon_\lambda^{\mu,*}(p) e^{ipx} \right) \tag{A.38}$$

and the ghost fields

$$c(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0} \left(c(\vec{p}) e^{-ipx} + c^\dagger(\vec{p}) e^{ipx} \right) |_{p^0=|\vec{p}|} \tag{A.39}$$

$$\bar{c}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0} \left(\bar{c}(\vec{p}) e^{-ipx} - \bar{c}^\dagger(\vec{p}) e^{ipx} \right) |_{p^0=|\vec{p}|} \tag{A.40}$$

The sum over the polarization vectors is over the two transverse polarizations with polarization vectors $\epsilon_\pm$ and two additional "longitudinal" and "scalar" polarizations given by

$$\epsilon_L^\mu(\vec{p}) = \begin{pmatrix} |\vec{p}| \\ \vec{p} \end{pmatrix} = p^\mu \qquad \epsilon_S^\mu(\vec{p}) = \frac{1}{2|\vec{p}|^2} \begin{pmatrix} |\vec{p}| \\ -\vec{p} \end{pmatrix} \tag{A.41}$$

The polarization vectors are normalized as

$$(\epsilon_\lambda(p) \cdot \epsilon^*_{\lambda'}(p)) = -\delta_{\lambda\lambda'}, \qquad \lambda, \lambda' = \pm \tag{A.42}$$

$$(\epsilon_S(p) \cdot \epsilon_L(p)) = 1 \tag{A.43}$$

$$(\epsilon_{S/L}(p) \cdot \epsilon_{S/L}(p)) = 0 \tag{A.44}$$

The BRST transformations of asymptotic fields in Feynman gauge ($\xi = 1$) are obtained by the limit $g_s \to 0$:

$$[\mathrm{i}Q_\mathrm{B}, A^{a,\mu}(x)] = \partial_\mu c^a(x) \tag{A.45}$$

$$\{\mathrm{i}Q_\mathrm{B}, c^a(x)\} = 0 \tag{A.46}$$

$$\{\mathrm{i}Q_\mathrm{B}, \bar{c}^a(x)\} = B^a(x) = -\partial_\mu A^{a,\mu}(x) \tag{A.47}$$

$$[\mathrm{i}Q_\mathrm{B}, B^a(x)] = 0 \tag{A.48}$$

Inserting the mode decomposition and comparing coefficients, one finds the transformations of the creation operators

$$[Q_\mathrm{B}, a^\dagger_\pm(\vec{p})] = [Q_\mathrm{B}, a^\dagger_S(\vec{p})] = 0 \tag{A.49}$$

$$[Q_\mathrm{B}, a^\dagger_L(\vec{p})] = c^a(\vec{p}) \tag{A.50}$$

$$\{Q_\mathrm{B}, c^\dagger(\vec{p})\} = 0 \tag{A.51}$$

$$\{Q_\mathrm{B}, \bar{c}^\dagger(\vec{p})\} = -a^\dagger_S(\vec{p}) \tag{A.52}$$

$$[Q_\mathrm{B}, a^\dagger_S(x)] = 0 \tag{A.53}$$

Since the vacuum satisfies $Q_\mathrm{B}\,|0\rangle = 0$ the states obtained by acting with the creation operator on the vacuum are classified as follows:

- BRST exact (zero-norm) states:

$$|c\rangle = Q_\mathrm{B}\,|g, L\rangle \tag{A.54}$$

$$|g, S\rangle = -Q_\mathrm{B}\,|c\rangle \tag{A.55}$$

$$\tag{A.56}$$

- BRST closed but not exact states:

$$|g, \pm\rangle \tag{A.57}$$

- physical states: equivalence classes of closed modulo exact states:

$$|g, \pm\rangle \sim |g, \pm\rangle + \alpha\ |g, S\rangle \tag{A.58}$$

The physical Hilbert space $\mathcal{H}_{\text{phys}}$ is usually defined as the equivalence class of the BRST-closed modulo exact states with ghost number zero. One can show [21] that because of the scalar products (A.43) and (A.44) the annihilation and creation operators of the unphysical modes satisfy the commutation relation

$$[a_S(\vec{k}), a_L^\dagger(\vec{p})] = \delta^3(\vec{k} - \vec{p}) \tag{A.59}$$

This implies that the S-matrix element for the state $|g, S\rangle$ is obtained by Feynman diagrams calculated with the polarization vector ϵ_L:

$$\langle \ldots A^\mu a_S^\dagger |0\rangle \to \langle \ldots |0\rangle\, \epsilon_L^\mu \tag{A.60}$$

Therefore the equivalence of the states (A.58) implies the equivalence of the polarization vectors

$$\epsilon_\pm^\mu \sim \epsilon_\pm^\mu + \alpha\epsilon_L^\mu = \epsilon_\pm^\mu + \alpha p^\mu, \tag{A.61}$$

i.e. the usual invariance under gauge transformations.

A.4 Consequences of BRST invariance

The BRST invariance is essential for the proofs of unitarity, gauge independence and renormalizability of gauge theories. To illustrate this, we briefly sketch how it can be used to show the gauge independence of S-matrix elements and to derive the Slavnov-Taylor identities.

A.4.1 Gauge independence

We have seen above that the gauge fixed Lagrangian has the form

$$\mathcal{L} = \mathcal{L}_0 + \delta_{\text{B}}\mathcal{F} \tag{A.62}$$

with

$$\mathcal{F} = \bar{c}^a (f^a + \frac{\xi}{2} B^a) \tag{A.63}$$

Consider a variation of the gauge-fixing functional

$$f^a[A] \to f^a[A] + \Delta f^a[A], \tag{A.64}$$

which implies the variation of the gauge fixing Lagrangian

$$\delta\mathcal{L} = \delta_{\mathrm{B}}(\bar{c}^a \delta f^a) = \{\mathrm{i}Q_{\mathrm{B}}, \bar{c}^a \Delta f^a[A]\}. \tag{A.65}$$

One can show that the change of a matrix element under the variation of the gauge fixing term is given by

$$\begin{aligned} \langle\phi_{\mathrm{phys}}|\psi_{\mathrm{phys}}\rangle_{f+\Delta f} - \langle\phi_{\mathrm{phys}}|\psi_{\mathrm{phys}}\rangle_f &= \int \mathrm{d}^4x\, \langle\phi_{\mathrm{phys}}|\mathrm{i}\delta\mathcal{L}(x)|\psi_{\mathrm{phys}}\rangle_f \\ &= \mathrm{i}\int \mathrm{d}^4x\, \langle\phi_{\mathrm{phys}}|\{\mathrm{i}Q_{\mathrm{B}}, \bar{c}^a(x)\Delta f^a[A(x)]\}|\psi_{\mathrm{phys}}\rangle \\ &= 0, \end{aligned} \tag{A.66}$$

which vanishes because of the definition of the physical states. For a more careful discussion including the LSZ reduction and renormalization see [21, 22].

A.4.2 Slavnov Taylor identities

We can derive the general **Slavnov Taylor Identities** of the theory by sandwiching the commutator (or anticommutator) of an arbitrary products of fields with the BRS-charge between physical fields:

$$\begin{aligned} 0 &= \langle\phi_{\mathrm{phys}}|\,\mathrm{T}[[\mathrm{i}Q_B, \Psi_1\Psi_2\ldots\Psi_n]_\pm]|\psi_{\mathrm{phys}}\rangle \\ &= \sum_i (-)^{s(i)}\,\langle\phi_{\mathrm{phys}}|\,\mathrm{T}[\Psi_1\ldots\delta_{\mathrm{B}}\Psi_i\ldots\Psi_n]|\psi_{\mathrm{phys}}\rangle \end{aligned} \tag{A.67}$$

As example consider identity obtained from the matrix element

$$\langle 0|\bar{c}^a(x)A^{\mu,b}(y)|Q,\bar{Q}\rangle_{\mathrm{phys}} \tag{A.68}$$

The STI implies

$$0 = \langle 0|(\delta_{\mathrm{B}}\bar{c}^a(x))A^{\mu,b}(y)|Q,\bar{Q}\rangle - \langle 0|\bar{c}^a(x)\delta_{\mathrm{B}}A^{\mu,b}(y)|Q,\bar{Q}\rangle \tag{A.69}$$

i.e.

$$\langle 0|\underbrace{B^a(x)}_{=-\partial_\mu A^a} A^{\mu,b}(y)|Q,\bar{Q}\rangle = \langle 0|\bar{c}^a(x)(\partial_\mu c^b + g_s f^{abc}c^b A^c_\mu)(y)|Q,\bar{Q}\rangle \tag{A.70}$$

At tree-level the bilinear term in the transformation of the gluon field does not contribute and one has the relation

$$\partial_{x,\mu}\,\langle 0|A^a_\mu(x)A^{\mu,b}(y)|Q,\bar{Q}\rangle = \partial_{y,\mu}\,\langle 0|\bar{c}^a(x)c^b(y)|Q\bar{Q}\rangle \tag{A.71}$$

Performing the LSZ reduction on the photon and ghost fields one obtains the identity (4.130)

$$\tilde{\mathcal{M}}_{\mu\nu}k_1^\mu = -k_{2,\nu}\mathcal{M}_{c_{k_1}\bar{c}_{k_2}} \tag{A.72}$$

found previously by an explicit calculation. Note that the ghost propagator connects anti-ghost and ghost fields so the LSZ reduction of an antighost field gives a ghost amplitude.

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