Revision Questions to Advanced Quantum Mechanics

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The purpose of these questions is to give you some orientation and to serve as starting points for discussions in the tutorials. They do not cover everything you should know.

1. What is the statement of Wigner's theorem?

2. What is the difference between a group and a representation thereof? Can we always reconstruct the group from a given representation?

3. What is the relation between a Lie group and a Lie algebra? Give an example.

4. A certain representation $\mathcal{U}$ of a group $\mathcal{G}$ assigns matrices of the form

$$\mathcal{U}(g) = \begin{pmatrix} \mathcal{A}(g) & \mathcal{B}(g) \\ 0_{k\times n} & \mathcal{C}(g) \end{pmatrix}$$

to the elements $g \in \mathcal{G}$, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are matrices and $0_{k\times n}$ the $k \times n$ zero matrix. Is there an invariant subspace? Is this representation reducible or irreducible? How would the matrices look like if the representation was fully reducible? The Hamiltonian $\hat{H}$ is invariant under the action of $\mathcal{G}$, and we know that the irreducible representations of the group are of dimension $d > 1$, what can we say about the energy spectrum of $\hat{H}$?

5. What happens to irreducible representations when enlarging or reducing the symmetry?

6. What is the statement of Schur's lemma?

7. Prove that all irreducible representations of Abelian groups have are 1-dimensional.

8. Prove that if a physical system, described by the Hamiltonian $\hat{H}$, is invariant under a symmetry transformation generated by the operator $\hat{A}$, then $[\hat{H}, \hat{A}] = 0$.

9. What are Wigner's D and d functions?

10. What is the statement of Bloch's theorem? What are Bloch waves?

11. What is the definition of an angular momentum $\vec{J}$ in quantum mechanics? If we write the eigenvalues of $\vec{J}^2$ as $\hbar^2 j(j+1)$, which conditions must j fulfil?

12. The orbital angular momentum $\vec{L}$ and the spin angular momentum $\vec{S}$ are the generators of which symmetry transformations? Can you write down the general form of the corresponding unitary operators? How are these transformations realised in the Hilbert space of spinor wave functions?

13. Let $\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$ be the sum of two angular momenta. What are the possible eigenvalues of $\vec{J}^2$? What can you say about the following commutators?

$$[\vec{J}^2, J_x], \qquad [\vec{J}^2, (\vec{J}^{(1)})^2], \qquad [\vec{J}^2, \vec{J}^{(1)} \cdot \vec{J}^{(2)}], \qquad [\vec{J}^2, J_z^{(1)}].$$

14. Can you solve the general eigenvalue problem of angular momentum?

15. What are Clebsch–Gordan coefficients? What is the Clebsch–Gordan series?

16. Add two angular momenta with quantum numbers $j_1, j_2 \leq 2$ of your choice and compare your results for the Clebsch–Gordan coefficients to those given in `http://pdg.lbl.gov/2018/reviews/rpp2018-rev-clebsch-gordan-coefs.pdf`.

17. How can operators be classified according to their behaviour under rotations? What are irreducible spherical tensors?

18. What does the Wigner–Eckart theorem state? What does it imply for scalar and vector operators when working in a subspace of dimension $2j+1$ of fixed angular momentum j?

19. Where is the WKB method applicable? How does it work?

20. How does Ritz's variational method work for excited states?

21. How does time-independent perturbation theory work? What are its assumptions? Are there subtleties?

22. What are the differences between Schrödinger, Heisenberg, and interaction picture?

23. Using first-order perturbation theory for a system with the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}'(t)$, we found that the transition probability from the state $|\varphi_i\rangle$ to the state $|\varphi_{f\neq i}\rangle$ is given by

$$W_{fi}(t, t_0) = \frac{1}{\hbar^2} \left| \int_{t_0}^{t} \mathrm{d}t' \, \mathrm{e}^{\mathrm{i}\omega_{fi}t'} \langle \varphi_f | \hat{H}'(t') | \varphi_i \rangle \right|^2 .$$

Explain all the elements in this expression.

24. Let us consider the transition rate

$$\frac{W_{fi}}{T} \underset{T\to\infty}{\sim} \frac{2\pi}{\hbar} |h_{fi}|^2 \delta(E_f - E_i - \hbar\omega) + \frac{2\pi}{\hbar} |h_{if}|^2 \delta(E_f - E_i + \hbar\omega).$$

Explain the meaning of all the terms and describe for which transitions and under which conditions this rule can be used. What is Fermi's golden rule?

25. What is the Hamiltonian of a charged particle (with and without spin) in an electromagnetic field?

26. How are Green's functions defined for the time(-in)dependent Schrödinger equation? Suppose you know an orthonormal basis of $\hat{H}$, how can you get the Green's functions?

27. Derive the Born series from the Lippman–Schwinger equation.

28. You want to calculate the elastic scattering cross section σ of a short-range central potential at low energies. Which method would you use and why? What if you aim for the high-energy behaviour of σ? Does your answer depend on the strength of the potential?

29. The scattering amplitude in first Born approximation is given by

$$f_k(\Omega) = -\frac{M}{2\pi\hbar^2} \int \mathrm{d}^3x \, \mathrm{e}^{\mathrm{i}(\vec{k}-\vec{k}')\cdot\vec{x}} \, V(\vec{x}).$$

Describe all quantities in this expression and the approximations made in its derivation.

30. What is the statement of the optical theorem? Can it be understood intuitively?