Exercises to Advanced Quantum Mechanics — Sheet 8
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Exercise 8.1 *Irreducible spherical tensors* (2 points)

a) Form an irreducible spherical tensor $T_m^{(3)}$ out of products $u_a v_b w_c$ of the real components u_a, v_b, w_c of the three cartesian vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

b) Proof that

$$Z_m^{(j)} = \sum_{m_1,m_2} X_{m_1}^{(j_1)} Y_{m_2}^{(j_2)} \langle j_1 j_2 m_1 m_2 | jm \rangle \tag{1}$$

is an irreducible spherical tensor operator of rank j if $X^{(j_1)}$ and $Y^{(j_2)}$ are both irreducible spherical tensors of ranks j_1 and j_2, respectively.

Hint: Make use of the formula from Exercise 7.3.

Exercise 8.2 *WKB method for s-states in central potentials* (2 points)

Consider a particle of mass m in a central potential $V(r)$ in 3 dimensions ($r = |\vec{x}|$). For vanishing angular momentum ($l = 0$), the wave function $\psi(\vec{x})$ is spherically symmetric and can be written as $\psi(\vec{x}) = u(r)/r$, where the radial function $u(r)$ plays the role of the wave function of the equivalent 1-dimensional problem with an effective potential $V_{\text{eff}}(r) = V(r)$ (no centrifugal term for $l = 0$). We are interested in bound states of energy E for which there is only one classical turning point at $r = r_E$ with $V(r_E) = E$, $V(r) < E$ for $r < r_E$, and $V'(r_E) > 0$. You may assume that the potential is finite at $r = 0$.

a) Partition the entire r range in appropriate regions and use the WKB method to construct approximate solutions $u(r)$ individually in each region. Which boundary or matching conditions must $u(r)$ satisfy in the classically allowed region?

b) Apply the boundary and matching conditions from a) to fix $u(r)$ in the classically allowed region. Show that the approximate energy eigenvalues satisfy the quantisation condition

$$\oint \mathrm{d}r\, p_r(r) = h\left(n + \frac{3}{4}\right), \qquad n = 0, 1, 2, \ldots, \tag{2}$$

where h is Planck's constant and p_r the radial momentum.

Please turn over!

Exercise 8.3 *Linear potential and WKB method* (2 points)

Consider a particle with mass m in a one-dimensional potential $V(x) = \varepsilon|x|$ with $\varepsilon > 0$.

a) Determine an approximation for the energy eigenvalues E_n ($n = 0, 1, 2, \dots$) using the WKB method.

b) Derive the antisymmetric wave functions $\psi(x) = -\psi(-x)$ upon using the results of Exercise 2.3 with the help of a symmetry argument. To which n-values of a) do these wave functions correspond? Compare the exact energy eigenvalues E_n with the respective approximations obtained in a) numerically.

   Hint: Take the zeroes of the Airy function from the literature.

Exercise 8.4 *Second-order perturbation theory – a delicate case* (2 points)

Consider a three-state system with the following Hamiltonian in matrix representation,

$$\hat{H} = \begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix}, \tag{3}$$

where a, b are considered as small perturbations ($|a|, |b| \ll |E_2 - E_1|$) and $E_{1,2}$ are the (real) energy eigenvalues of the unperturbed system.

a) Calculate the exact energy eigenvalues of the system and expand them in the small quantities a, b to the first non-trivial order.

b) Calculate the energy eigenvalues using second-order perturbation theory.