Exercises to Advanced Quantum Mechanics — Sheet 7
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Exercise 7.1 *Landau levels reloaded* (4 points)

We reconsider Exercise 6.1, where an electron (electric charge $q = -e$) is put into a homogeneous magnetic field aligned along the x_3 axis ($\vec{B} = \nabla \times \vec{A} = B\vec{e}_3$ with the convenient choice $\vec{A} = \frac{1}{2}\vec{B} \times \vec{x}$ for the vector potential $\vec{A}$ and $B > 0$). Our aim is to construct simultaneous eigenstates of the Hamiltonian $\hat{H}$ and the orbital angular momentum component $\hat{L}_3$ ($[\hat{H}, \hat{L}_3] = 0$). Since the electron spin and the movement in the x_3 direction are not touched by this issue, we ignore spin effects and the x_3-dependence in the following.

a) The part $\hat{H}_{12}$ of the Hamiltonian relevant for the movement in the x_1-x_2-plane can be written as

$$\hat{H}_{12} = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right), \qquad \omega = \frac{eB}{m}, \tag{1}$$

with the shift operators

$$a = \frac{1}{\sqrt{2\hbar\omega m}} \left(\hat{\Pi}_1 - \mathrm{i}\hat{\Pi}_2 \right) \tag{2}$$

and $a^\dagger$ constructed from $\hat{\Pi}_j = \hat{p}_j + eA_j(\hat{\vec{x}})$, $j = 1, 2$. Verify the form (1) of $\hat{H}_{12}$ and that the operators $a, a^\dagger$ obey the usual commutator relations of a harmonic oscillator.

b) The operators $\hat{X}_1 = \hat{x}_1 - \frac{1}{eB}\hat{\Pi}_2$ and $\hat{X}_2 = \hat{x}_2 + \frac{1}{eB}\hat{\Pi}_1$ represent the coordinates of the (classical) center of the cyclotron motion. Show that

$$b = \sqrt{\frac{eB}{2\hbar}} (\hat{X}_2 - \mathrm{i}\hat{X}_1) \tag{3}$$

and $b^\dagger$ are the shift operators of another harmonic oscillator which is independent from the one defined by (2).

c) Express the angular momentum operator $\hat{L}_3$ in terms of a, $a^\dagger$ and b, $b^\dagger$. Let $|n, m_3\rangle$ denote the common eigenstates of $\hat{H}_{12}$ and $\hat{L}_3$ with $\hat{H}_{12}|n, m_3\rangle = \hbar\omega(n + \frac{1}{2})|n, m_3\rangle$ and $\hat{L}_3|n, m_3\rangle = \hbar m_3|n, m_3\rangle$. What are the possible quantum numbers m_3 for any given $n \in \mathbb{N}_0$? How can $|n, m_3\rangle$ be constructed from $|0, 0\rangle$ for any allowed n, m_3?

Hint: Find the eigenvalues of $b^\dagger b$ first.

d) Derive the position space wave function $\psi_{n,m_3}(\vec{x}) = \langle \vec{x}|n, m_3\rangle$ of the ground states $|0, m_3\rangle$ in cylindrical coordinates $\vec{x} = (\rho\cos\phi, \rho\sin\phi, x_3)^T$. Show that the condition that the wave functions must be normalisable leads to the same restrictions on the quantum number m_3 for the ground states as found in c).

Please turn over!

Exercise 7.2 *Addition of angular momenta –* $D^{(1/2)} \otimes D^{(1/2)} \otimes D^{(1/2)}$ (3 points)

Consider a quantum-mechanical system consisting of three spin-$\frac{1}{2}$ particles, ignoring all degrees of freedom other than spin. Labelling the respective spin parts of the one-particle states by $|\uparrow\rangle_k \equiv |\frac{1}{2}, \frac{1}{2}\rangle_k$, $|\downarrow\rangle_k \equiv |\frac{1}{2}, -\frac{1}{2}\rangle_k$ for particle $k = 1, 2, 3$, construct linear combinations of the product states $|\uparrow\uparrow\uparrow\rangle \equiv |\uparrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3$, etc. that are simultaneous eigenstates of $\vec{J}^{\,2}$ and J_3, where $\vec{J} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$ is the total spin of the system. How is the product representation $D^{(1/2)} \otimes D^{(1/2)} \otimes D^{(1/2)}$ expressed in terms of a direct sum of irreducible representations?

Exercise 7.3 *Recursion relation for Clebsch–Gordan coefficients* (2 points)

We consider a quantum-mechanical system consisting of two parts that are each described by angular momentum eigenstates $|j_k, m_k\rangle$ $(k = 1, 2)$ of $\vec{J}_k^{\,2}$ and $J_{k,3}$ of the respective angular momentum operators $\vec{J}_k$:

$$\begin{aligned} \vec{J}_k^{\,2} |j_k, m_k\rangle &= \hbar^2 j_k (j_k + 1) |j_k, m_k\rangle, \qquad && j_k = 0, \tfrac{1}{2}, 1, \ldots, \\ J_{k,3} |j_k, m_k\rangle &= \hbar m_k |j_k, m_k\rangle, && m_k = -j_k, -j_k + 1, \ldots, j_k. \end{aligned}$$

The transition from the basis of product states $|j_1, m_1; j_2, m_2\rangle \equiv |j_1, m_1\rangle |j_2, m_2\rangle$ to the basis $|j, m\rangle$ of eigenstates of $\vec{J}^{\,2}$ and J_3 of the total angular momentum $\vec{J}$ is described in terms of Clebsch–Gordan coefficients $\langle j_1, m_1; j_2, m_2 | j, m\rangle$:

$$|j, m\rangle = \sum_{\substack{m_1, m_2 \\ m = m_1 + m_2}} |j_1, m_1; j_2, m_2\rangle \, \langle j_1, m_1; j_2, m_2 | j, m\rangle. \tag{4}$$

With the help of the shift operators $J_\pm = J_{1\pm} + J_{2\pm}$ derive the following recursion relations for the Clebsch–Gordan coefficients:

$$\begin{aligned} &\sqrt{j(j+1) - m(m-1)} \, \langle j_1, m_1; j_2, m_2 | j, m-1\rangle \\ &\qquad = \sqrt{j_1(j_1+1) - m_1(m_1+1)} \, \langle j_1, m_1 + 1; j_2, m_2 | j, m\rangle \\ &\qquad + \sqrt{j_2(j_2+1) - m_2(m_2+1)} \, \langle j_1, m_1; j_2, m_2 + 1 | j, m\rangle. \end{aligned}$$