

Exercises to Advanced Quantum Mechanics — Sheet 6

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Exercise 6.1 *Landau levels of electrons in a magnetic field* (2 points)

Consider an electron (electric charge $q = -e$) in a homogeneous magnetic field, which is aligned along the x_3 axis ($\vec{B} = \nabla \times \vec{A} = B\vec{e}_3$).

- a) Generate the Hamilton operator upon applying the “minimal substitution” $\hat{p} \rightarrow \hat{\Pi} = \hat{p} - q\vec{A}(\hat{x})$ to the Hamilton operator of a free particle and subsequently adding the interaction part $\hat{H}_s = -\vec{\mu} \cdot \vec{B}$ of the magnetic field \vec{B} with the magnetic moment $\vec{\mu} = \frac{gq}{2m_e}\vec{S}$ induced by the spin \vec{S} .
- b) Reduce the eigenvalue problem of the Hamilton operator to appropriate one-dimensional problems for the spatial motion and show that the eigenvalues have the form

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \hbar\omega \left(n + \frac{1}{2} \right) + g_e \hbar\omega_L m_s, \quad n \in \mathbb{N}_0, \quad m_s = \pm \frac{1}{2}, \quad (1)$$

where $\hbar k$ is the continuous eigenvalue of \hat{p}_3 , m_s corresponds to the spin orientation, and $g_e = 2.002\dots$ denotes the g -factor of the positron.

Exercise 6.2 *SU(2) matrices and rotations* (2 points)

SU(2) is the Lie group of dimension 3 consisting of all complex, unitary 2×2 matrices A with $\det A = 1$. A convenient way to parametrise A is in terms of a rotation angle θ ($0 \leq \theta \leq 2\pi$) and a 3-dim. real unit vector \vec{e} (e.g. parametrised by its polar and azimuthal angles ϑ and φ , respectively), defining the “rotation vector” $\vec{\theta} = \theta\vec{e}$:

$$A(\vec{\theta}) = \exp \left\{ -\frac{i}{2} \vec{\theta} \cdot \vec{\sigma} \right\} = \cos \frac{\theta}{2} \mathbb{1} - i(\vec{e} \cdot \vec{\sigma}) \sin \frac{\theta}{2}, \quad (2)$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the “vector” formed by the Pauli matrices σ_a .

- a) Show that all $A \in \text{SU}(2)$ can be expressed in terms of the form (2). Express θ and \vec{e} in terms of the coefficients of A .
- b) Associating a matrix $V = \vec{v} \cdot \vec{\sigma}$ with each 3-dim. vector \vec{v} , show that the transformation $V \rightarrow V' = AVA^\dagger$ rotates the vector \vec{v} into the vector $\vec{v}' = R(\vec{\theta})\vec{v}$ corresponding to the matrix $V' = \vec{v}' \cdot \vec{\sigma}$, where $R(\vec{\theta})$ is the general rotation given in Exercise 5.2. Which matrices A correspond to a given rotation in 3-dimensional space?

Please turn over!

Exercise 6.3 *Dynamical symmetry of the isotropic 3-dim. harmonic oscillator* (4 points)

In Exercise 2.1 you have decomposed the Hamiltonian \hat{H} of the isotropic 3-dimensional harmonic oscillator into its individual parts corresponding to cartesian coordinates, resulting in

$$\hat{H} = \hbar\omega \sum_{j=1}^3 \left(a_j^\dagger a_j + \frac{1}{2} \right), \quad (3)$$

where a_j and a_j^\dagger ($j = 1, 2, 3$) are the usual shift operators for the movement in x_i direction obeying the relations

$$[a_j, a_k] = 0, \quad [a_j^\dagger, a_k^\dagger] = 0, \quad [a_j, a_k^\dagger] = \delta_{jk}. \quad (4)$$

- For which complex matrices U does the replacement $\vec{a} = (a_1, a_2, a_3)^T \rightarrow \vec{a}' = U\vec{a}$ represent a symmetry? The set of all U defines a Lie group. What is the dimension of this group? How are the generators X_a of this group characterised?
- What is the role of the one-dimensional subgroup consisting of pure phase transformations $U = e^{i\theta}\mathbb{1}$? What is an appropriate condition on the matrices U of a) to eliminate those phase transformations?
- What is the relation of the symmetry transformations represented by U defined in a) to rotations in 3-dim. space? What is the relation between the X_a and orbital angular momentum? Identify the “accidental symmetry” that goes beyond pure rotational invariance.
- The energy eigenstates to the eigenvalue $E_N = \hbar\omega(N + \frac{3}{2})$ with a fixed number $N \in \mathbb{N}_0$ are proportional to the states $a_{j_1}^\dagger \dots a_{j_N}^\dagger |0\rangle$, which transform under U like components of a symmetric tensor of rank N in three dimensions:

$$a_{j_1}^\dagger \dots a_{j_N}^\dagger |0\rangle \rightarrow a_{j'_1}^\dagger \dots a_{j'_N}^\dagger |0\rangle = \sum_{k_1, \dots, k_N} U_{j_1 k_1}^* \dots U_{j_N k_N}^* a_{k_1}^\dagger \dots a_{k_N}^\dagger |0\rangle. \quad (5)$$

Deduce the degree of degeneracy of E_N from this consideration.