**Exercises to Advanced Quantum Mechanics — Sheet 2**
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**Exercise 2.1** *Three-dimensional isotropic harmonic oscillator* (3 points)

The Hamilton operator of the three-dimensional isotropic oscillator is

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + \frac{m\omega^2}{2}\,\hat{\vec{x}}^2, \tag{1}$$

with $\hat{\vec{x}}$ and $\hat{\vec{p}}$ denoting the cartesian position and momentum operators of a particle of mass $m$, respectively.

a) Which are the conserved quantities of the system?

b) Use your knowledge about the one-dimensional harmonic oscillator and its eigenstates to determine the energy spectrum and the corresponding eigenstates of the three-dimensional system by separating the cartesian variables.

c) What is the degree of degeneracy of the energy levels?

**Exercise 2.2** *Simplified version of Wigner's theorem* (2 points)

a) Show that a linear operator $U$ that preserves all norms of states in a Hilbert space, i.e.

$$\|\psi\| = \|U\psi\| \qquad \text{for all} \qquad |\psi\rangle \in \mathcal{H}, \tag{2}$$

is unitary.

b) Analogously, show that an antilinear operator $U$ that satisfies the condition (2), where antilinearity means

$$U\Big(a|\psi\rangle + b|\phi\rangle\Big) = a^* U|\psi\rangle + b^* U|\phi\rangle, \qquad |\psi\rangle, |\phi\rangle \in \mathcal{H}, \quad a, b \in \mathbb{C}, \tag{3}$$

is antiunitary, i.e. $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle^*$. Note that the adjoint of an antiunitary operator $U$ is defined by $\langle U\phi|\psi\rangle = \langle\phi|U^\dagger\psi\rangle^*$.

*Comment:* Wigner more generally showed that the requirement $|\langle U\phi|U\psi\rangle| = |\langle\phi|\psi\rangle|$ for all $|\phi\rangle, |\psi\rangle$ implies that $U$ is either unitary (which implies linearity) or antiunitary (which implies antilinearity). A complete proof can be found in S. Weinberg, *The Quantum Theory of Fields*, Vol. I, p. 91.

*Please turn over!*

**Exercise 2.3** *Particle in a homogenous electric field* (4 points)

Consider a particle of mass $m$ and electric charge $q$ in a homogenous electric field of field strength $\mathcal{E}$ in $x$ direction. You need not consider the motion in the $y$ and $z$ directions in the following.

a) Formulate the time-independent Schrödinger equation in momentum representation for a fixed energy $E$. Determine the wave function $\langle k|E\rangle = \tilde{\psi}_E(k)$ of the energy eigenstate $|E\rangle$, where $k$ is the usual wave number and $\tilde{\psi}_E(k)$ is related to the wave function $\langle x|E\rangle = \psi_E(x)$ in position space as follows,

$$\tilde{\psi}_E(k) \;=\; \int_{-\infty}^{+\infty} \mathrm{d}x\, \langle k|x\rangle\, \langle x|E\rangle \;=\; \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx}\, \psi_E(x). \tag{4}$$

Which values for $E$ are allowed?

b) Calculate the wave function $\psi_E(x)$ in position space from $\tilde{\psi}_E(k)$ upon inverting Eq. (4) and express the result in terms of the Airy function

$$\mathrm{Ai}(\xi) \;=\; \frac{1}{\pi}\int_0^\infty \mathrm{d}u\, \cos\left(\frac{u^3}{3} + \xi u\right)\,, \qquad \xi \in \mathbf{R}. \tag{5}$$

Normalize the wave functions according to

$$\langle E'|E\rangle \;=\; \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi_{E'}(x)^*\psi_E(x) \;=\; \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi}\, \tilde{\psi}_{E'}(k)^*\tilde{\psi}_E(k) \;=\; \delta(E-E'). \tag{6}$$

c) Sketch $\psi_E(x)$ for a positive charge $q$ graphically, making use of the asymptotics

$$\mathrm{Ai}(\xi) \underset{\xi\to+\infty}{\widetilde{\qquad}} \frac{\mathrm{e}^{-\frac{2}{3}\xi^{3/2}}}{2\sqrt{\pi}\,\xi^{1/4}}, \qquad \mathrm{Ai}(\xi) \underset{\xi\to-\infty}{\widetilde{\qquad}} \frac{\sin(\frac{2}{3}(-\xi)^{3/2} + \frac{1}{4}\pi)}{\sqrt{\pi}\,(-\xi)^{1/4}}. \tag{7}$$

Make sure that the graph respects the sign of the curvature given by $\psi_E''(x)/\psi_E(x) = \frac{2m}{\hbar^2}(V-E) \lessgtr 0$ in the allowed and forbidden regions.

d) For the time $t=0$ the particle is described by the wave function

$$\psi(x,0) \;=\; \left(2\pi\sigma^2\right)^{-1/4} \exp\left\{-\frac{x^2}{4\sigma^2} + \mathrm{i}k_0 x\right\} \tag{8}$$

with some real constants $\sigma$ and $k_0$. Calculate the expectation values $\langle\hat{x}\rangle_\psi$ and $\langle\hat{p}\rangle_\psi$ for any time $t$. What is the meaning of $\sigma$ and $k_0$?

*Hint:* Ehrenfest's theorem may save you from a lengthy calculation.