Exercise 19 *Auxiliary relations for conventional gauge transformations* (3 points)

A non-abelian local gauge transformation can represented using the unitary matrix

$$U\left(\omega^A(x)\right) = \exp\left\{-igT^A\omega^A(x)\right\}, \tag{1}$$

where T^A denotes the generators of the gauge group and ω^A are the group parameters that are real functions of space-time x^μ. Show the relation

$$\exp\left\{-igT^A\omega^A(\bar{y})\right\} = U + i(\theta\bar{\sigma}^\mu\bar{\theta})(\partial_\mu U) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})(\Box U), \tag{2}$$

which is an expansion of the generalized exponential as a function of coordinates $\bar{y}^\mu = x^\mu + i\theta\bar{\sigma}^\mu\bar{\theta}$. The exponentials on the r.h.s. of Eq. (2) are functions of x, as defined in Eq. (1).

Exercise 20 *Superfield of the non-abelian field-strength tensor* (8 points)

In Wess–Zumino gauge the components of a vector superfield of a non-abelian gauge symmetry reads

$$V^A(x,\theta,\bar{\theta})\Big|_{\mathrm{WZ}} = -(\theta\bar{\sigma}^\mu\bar{\theta})A^A_\mu(x) - i(\theta\theta)\bar{\theta}\bar{\lambda}^A(x) + i(\bar{\theta}\bar{\theta})\theta\lambda^A(x) - \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D^A(x), \tag{3}$$

with $A = 1,\dots,\dim(G)$. The corresponding (SUSY gauge-covariant) superfield of the non-abelian field-strength tensor is defined as

$$\begin{aligned} TW_a(x,\theta,\bar{\theta}) &\equiv T^A W^A_a(x,\theta,\bar{\theta}) \\ &= \frac{1}{8g}(\bar{\mathcal{D}}\bar{\mathcal{D}})\exp\{2gTV(x,\theta,\bar{\theta})\}\mathcal{D}_a\exp\{-2gTV(x,\theta,\bar{\theta})\}, \end{aligned} \tag{4}$$

with T^A being the generators of the gauge group G which satisfy $[T^A,T^B] = iC^{ABC}T^C$. Calculate the component form of TW_a in terms of the coordinates $y^\mu = x^\mu - i\theta\bar{\sigma}^\mu\bar{\theta}$, $\eta = \theta$, and $\bar{\eta} = \bar{\theta}$:

$$TW_a(y,\eta) = T\lambda_a(y) + i\eta_a(TD(y)) - i(\eta\eta)\bar{\sigma}^\mu_{a\dot{b}}\left(D_{\mathrm{adj},\mu}\bar{\lambda}^{\dot{b}}(y)\right) + \frac{1}{2}\bar{\sigma}^\mu_{a\dot{b}}\sigma^{\dot{b}c,\nu}\eta_c\left(TF_{\mu\nu}(y)\right). \tag{5}$$

The matrix $D_{\mathrm{adj},\mu}$ denotes covariant derivative in adjoint representation (of G),

$$(D_{\mathrm{adj},\mu})^{BC} = \delta^{BC}\partial_\mu + gC^{ABC}A^A_\mu, \tag{6}$$

and $F^A_{\mu\nu}$ are the components of the non-abelian field-strength tensor,

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - gC^{ABC}A^B_\mu A^C_\nu. \tag{7}$$