Exercise 10 *Super Poincaré transformations* (6 points)

An important class of super Poincaré transformations (the class without the Lorentz transformations) is given as

$$U(a,\alpha,\bar{\alpha}) \;=\; \exp\left\{iaP + i\alpha Q + i\bar{\alpha}\bar{Q}\right\} \;\equiv\; \exp\left\{ia^\mu P_\mu + i\alpha^a_r Q_{a,r} + i\bar{\alpha}_{\dot{a},r}\bar{Q}^{\dot{a}}_r\right\}, \qquad (1)$$

where a^μ is a constant four-vector and α^a_r ($a = 1,2$; $r = 1,\dots,N$), as well as $\bar{\alpha}_{\dot{a},r} = (\alpha_{a,r})^*$, anticommuting spinor-like parameters. P, Q, and $\bar{Q}$ are the usual generators of the relativistic SUSY algebra (graded with grade N). We assume that the central charges vanish, i.e. $Z_{rs} = 0$.

a) Show that

$$U(a,\alpha,\bar{\alpha})^\dagger \;=\; U(-a,-\alpha,-\bar{\alpha}) \;=\; U(a,\alpha,\bar{\alpha})^{-1}. \qquad (2)$$

b) Prove the law of composition,

$$U(a^\mu,\alpha,\bar{\alpha})\,U(a'^\mu,\alpha',\bar{\alpha}') \;=\; U(a^\mu + a'^\mu + i\alpha\bar{\sigma}^\mu\bar{\alpha}' + i\bar{\alpha}\sigma^\mu\alpha', \alpha+\alpha', \bar{\alpha}+\bar{\alpha}'), \qquad (3)$$

where $\alpha\bar{\sigma}^\mu\bar{\alpha}' = \alpha^a_r\bar{\sigma}^\mu_{a\dot{b}}\bar{\alpha}'^{\dot{b}}_r$ and $\bar{\alpha}\sigma^\mu\alpha' = \bar{\alpha}^{\dot{a}}_r\sigma^\mu_{\dot{a}b}\alpha'^b_r$.
(Hint: First consider the commutators of aP, αQ, $\bar{\alpha}\bar{Q}$, $a'P$, $\alpha' Q$, $\bar{\alpha}'\bar{Q}$ and make use of the Baker–Campbell–Hausdorff formula afterwards.)

c) Is the product of U's commutative? Is it associative?

Exercise 11 *SUSY Oscillator* (6 points)

The Hamilton operators

$$H_B = b^+b^- + \tfrac{1}{2}, \qquad H_F = f^+f^- + \varepsilon \qquad (4)$$

together with the creation and annihilation operators $b^\pm$ and $f^\pm$,

$$[b^\pm, b^\pm] = 0, \quad [b^-, b^+] = 1, \qquad \{f^\pm, f^\pm\} = 0, \quad \{f^-, f^+\} = 1, \qquad (5)$$

define Bose- and Fermi oscillators. The parameter ε shall initially be an arbitrary constant. Bose- and Fermi oscillators can be combined into a SUSY oscillator as follows:

$$H = H_B + H_F \qquad \text{with} \quad [b^\pm, f^\pm] = [b^\pm, f^\mp] = 0. \qquad (6)$$

a) The energy eigensystem of the Bose and Fermi oscillators are given as

$$H_B|m\rangle_B = (m + \tfrac{1}{2})|m\rangle_B, \qquad |m\rangle_B = \frac{(b^+)^m}{\sqrt{m!}}|0\rangle_B, \qquad m = 0, 1, 2, \ldots, \tag{7}$$

$$H_F|n\rangle_F = (n + \varepsilon)|n\rangle_F, \qquad |1\rangle_F = f^+|0\rangle_F, \qquad n = 0, 1, \tag{8}$$

where the ground states are characterized by $b^-|0\rangle_B = f^-|0\rangle_F = 0$.
Determine the energy eigenstates of the SUSY oscillator by considering the product states $|\ldots\rangle = |\ldots\rangle_B|\ldots\rangle_F$. What is the degree of degeneracy of the eigenstates?

b) Calculate all (anti-)commutators of H and

$$Q^+ = b^- f^+, \qquad Q^- = (Q^+)^\dagger = b^+ f^-. \tag{9}$$

For which value of ε does H and $Q^\pm$ constitute a SUSY algebra?

c) How do the operators $b^\pm$, $f^\pm$, and $Q^\pm$ act on the energy eigenstates of the SUSY oscillator? Illustrate this in the (m, n)-plane, where m and n are defined in Eqs. (7) and (8).