Exercise 3.1 (3 points) *Bianchi identity of the Yang-Mills theory*

Consider a Yang-Mills theory with gauge fields A^a_μ, group generators T^a, and structure constants C^{abc}. The covariant derivative $D_\mu = \partial_\mu + igT^aA^a_\mu$ fulfills the Jacobian identity

$$[D_\mu, [D_\nu, D_\lambda]] + \text{ cycl. in } (\mu, \nu, \lambda) \;=\; 0.$$

Using this relation, derive the *Bianchi identity*

$$D^{ab}_\mu \tilde{F}^{b,\mu\nu} \;=\; 0$$

for the dual field-strength tensor $\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{a,\rho\sigma}$ where the field-strength tensor is defined as $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gC^{abc}A^b_\mu A^c_\nu$, and $D^{ab}_\mu = \delta^{ab}\partial_\mu + gC^{abc}A^c_\mu$ denotes the covariant derivative in the adjoint representation.

[Hint: First, prove that $D^{ab}_\mu F^b_{\nu\lambda} +$ cyclic in $(\mu, \nu, \lambda) \;=\; 0$.]

Exercise 3.2 (3 points) *Gauge-boson propagator*

The propagator $D^{ab}_{\mu\nu}(x)$ of the gauge field $A^a_\mu(x)$ is defined by

$$\left[g^{\mu\nu}\partial^2 + \left(\frac{1}{\xi} - 1\right)\partial^\mu\partial^\nu\right] D^{ab}_{\nu\rho}(x) \;=\; \delta^{ab}\,\delta^\mu_\rho\,\delta(x).$$

Upon inserting

$$D^{ab}_{\mu\nu}(x) \;=\; \int \frac{d^4q}{(2\pi)^4}\,\exp\{-iqx\}\,\tilde{D}^{ab}_{\mu\nu}(q)$$

into this definition, calculate the Fourier transform $\tilde{D}^{ab}_{\mu\nu}(q)$ of the propagator.

[Hint: The general ansatz $\tilde{D}^{ab}_{\mu\nu}(q) = \delta^{ab}\left[f_1(q^2)g_{\mu\nu} + f_2(q^2)q_\mu q_\nu\right]$ leads to a linear system of equations for f_1 and f_2.]

Exercise 3.3 (4 points) *Generating functional for the free charged scalar field*

The dynamics of the real scalar fields $\phi_k(x)$ $(k = 1, 2)$ describing free spin-0 bosons with mass m are determined by the Lagrangian $\mathcal{L}_{k,0} = -\frac{1}{2}\phi_k(\partial^2 + m^2)\phi_k$. The corresponding generating functionals for the Green's functions are given as

$$\begin{aligned}
Z_{k,0}[J_k] \;&=\; N_k \int \mathcal{D}\phi_k\,\exp\left\{i\int d^4x\,[\mathcal{L}_{k,0}(x) + J_k(x)\phi_k(x)]\right\}\\
&=\; \exp\left\{\frac{1}{2}\int d^4x\int d^4x'\;iJ_k(x)\,i\Delta_F(x-x')\,iJ_k(x')\right\}.
\end{aligned}$$

a) For both of the given forms of $Z_{k,0}[J_k]$, express the complete generating functional $Z_0[J^+, J^-] = Z_{1,0}[J_1]Z_{2,0}[J_2]$ by the following variables,

$$\phi^\pm \;=\; (\phi_1 \mp i\phi_2)/\sqrt{2}, \qquad J^\pm \;=\; (J_1 \mp iJ_2)/\sqrt{2}.$$

b) Calculate the Green's functions $G_0^{\phi^\pm\phi^\pm}(x_1, x_2)$ and $G_0^{\phi^\pm\phi^\mp}(x_1, x_2)$ from $Z_0[J^+, J^-]$ by taking functional derivatives.