Exercise 4.1 (4 points) *Massive gauge-boson propagator in R_ξ gauge*

The propagator $D_\xi^{\mu\nu}(x)$ of a massive gauge boson with mass M is defined by

$$\left[g_{\mu\nu}\left(\partial^2+M^2\right)+\left(\frac{1}{\xi}-1\right)\partial_\mu\partial_\nu\right]D_\xi^{\nu\rho}(x)=\delta_\mu^\rho\delta(x)\;.$$

a) Calculate the Fourier-transformed $\tilde{D}_\xi^{\mu\nu}(q)$ of the propagator by inserting

$$D_\xi^{\mu\nu}(x)=\int\frac{d^4q}{(2\pi)^4}\exp\{iqx\}\tilde{D}_\xi^{\mu\nu}(q)$$

into the differential equation given above. Make use of the decomposition of $\tilde{D}_\xi^{\mu\nu}(q)$ into transverse and longitudinal parts, $\tilde{D}_{T,\xi}(q)$ and $\tilde{D}_{L,\xi}(q)$, respectively, with

$$\tilde{D}_\xi^{\mu\nu}(q)=\tilde{D}_{T,\xi}(q)\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)+\tilde{D}_{L,\xi}(q)\frac{q^\mu q^\nu}{q^2}\;.$$

Determine $\tilde{D}_\xi^{\mu\nu}(q)$ in the limits $\xi\to 0$, $\xi\to 1$, and $\xi\to\infty$.

b) Given the generating functional

$$Z_0[J_\mu]=\frac{1}{N}\int\mathcal{D}A^\mu\exp\left\{i\int d^4x\left[\mathcal{L}_0+J_\mu A^\mu\right]\right\}$$

with

$$Z_0[0]=1\;,\quad\mathcal{L}_0=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}M^2A_\mu A^\mu-\frac{1}{2\xi}\left(\partial A\right)^2,$$

show that

$$Z_0[J_\mu]=\exp\left\{+\frac{1}{2}\int d^4x\int d^4x'\,iJ_\mu(x)iD_\xi^{\mu\nu}(x-x')iJ_\nu(x')\right\}\;.$$

Please turn over!

Exercise 4.2 (3 points) *Two-point Green's function of the ϕ^4 theory*

The interaction part of the Lagrangian of a ϕ^4 theory with a single, real scalar field ϕ is given as

$$\mathcal{L}_I = -\frac{g}{4!}\phi^4 \,.$$

Starting from the generating functionals, calculate the Green's function $G^{\phi\phi}(x_1, x_2)$ and the connected Green's function $G^{\phi\phi}_{\text{con}}(x_1, x_2)$ up to order $\mathcal{O}(g)$ and draw diagrams representing the resulting terms.

Exercise 4.3 (3 points) *Equation of motion for Green's functions*

Consider a quantum field theory of a real scalar field $\phi(x)$ with the Lagrangian $\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_I(\phi)$ where the free part is given by $\mathcal{L}_0(\phi) = -\frac{1}{2}\phi(\partial^2 + m^2)\phi$ and the interaction part $\mathcal{L}_I(\phi)$ is not further specified.

a) Verify explicitly that the free generating functional

$$Z_0[J] = \exp\Big\{+\frac{1}{2}\int d^4x \int d^4x'\, iJ(x) i\Delta_F(x-x') iJ(x')\Big\}$$

fulfills the following equation of motion

$$\left[\frac{\delta\mathcal{L}_0}{\delta\phi}\left(\frac{\delta}{i\delta J(x)}\right) + J(x)\right] Z_0[J] = 0.$$

b) Starting with this equation, derive the equations of motion for the free two- and four-point functions, $G_0^{\phi\phi}(x_1, x_2)$ and $G_0^{\phi\phi}(x_1, x_2, x_3, x_4)$ by taking the functional derivative.

c) By explicitly inserting the generating functional

$$Z[J] = \exp\left\{i\int d^4y\, \mathcal{L}_I\left(\frac{\delta}{i\delta J(x)}\right)\right\} Z_0[J] \,,$$

show that in a theory with interactions the following equation of motions hold,

$$\left[\frac{\delta\mathcal{L}}{\delta\phi}\left(\frac{\delta}{i\delta J(x)}\right) + J(x)\right] Z[J] = 0.$$

Use (and prove) the commutator relation

$$\left[\exp\left\{i\int d^4y\, \mathcal{L}_I\left(\frac{\delta}{i\delta J(x)}\right)\right\}, J(x)\right] = \frac{\delta\mathcal{L}_I}{\delta\phi}\left(\frac{\delta}{i\delta J(x)}\right) \exp\left\{i\int d^4y\, \mathcal{L}_I\left(\frac{\delta}{i\delta J(x)}\right)\right\}.$$