Exercises to Relativistic Quantum Field Theory — Sheet 9

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Exercise 9.1 *Relation between the Lorentz group and* $\mathrm{SL}(2,\mathbb{C})$ (1 point)

The group $\mathrm{SL}(2,\mathbb{C})$ consists of all complex 2×2 matrices A with $\det(A)=1$. Assign to each four-vector x^μ a 2×2 matrix $X = x_\mu\sigma^\mu$ and $\bar X = x_\mu\bar\sigma^\mu$ where $\sigma^\mu = (\mathbb{1},\sigma^1,\sigma^2,\sigma^3)$ and $\bar\sigma^\mu = (\mathbb{1},-\sigma^1,-\sigma^2,-\sigma^3)$. The matrices σ^μ and $\bar\sigma^\mu$ satisfy the relation $\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) = 2g^{\mu\nu}$.

a) Show that the inverse of the above assignment is given by

$$x^\mu = \frac{1}{2}\,\mathrm{Tr}(X\bar\sigma^\mu) = \frac{1}{2}\,\mathrm{Tr}(\bar X\sigma^\mu).$$

b) What is the meaning of $\det(X)$ and $\det(\bar X)$?

c) For two arbitrary matrices A, B of $\mathrm{SL}(2,\mathbb{C})$, show that the mappings $X\to X' = AXA^\dagger$ and $\bar X\to\bar X' = B\bar X B^\dagger$ define Lorentz transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

Hint: Consider the determinants.

d) How are the matrices A, B of c) and the matrices Λ_L and Λ_R of the fundamental representations of Exercise 8.2 related?

Hint: $\Lambda_\mathrm{R}^\dagger\sigma^\mu\Lambda_\mathrm{R} = \Lambda^\mu{}_\nu\sigma^\nu$, $\Lambda_\mathrm{L}^\dagger\bar\sigma^\mu\Lambda_\mathrm{L} = \Lambda^\mu{}_\nu\bar\sigma^\nu$.

Exercise 9.2 *Relations for Dirac matrices* (1.5 points)

The Dirac matrices γ_μ and γ_5 are defined by

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1},\qquad \gamma_0\gamma^\mu\gamma_0 = (\gamma^\mu)^\dagger,\qquad \gamma_5 = \gamma^5 = -\frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3.$$

The matrix γ_5 satisfies the relations $\{\gamma^\mu,\gamma^5\} = 0$ and $(\gamma_5)^\dagger = \gamma_5$. In the chiral basis, γ_5 has the representation

$$\gamma_5 = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}.$$

a) Calculate the following traces:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu),\qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma).$$

b) Prove the following trace relations:

$$\mathrm{Tr}(\gamma_5) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma_5) = 0,\qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma},$$

$$\mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}) = \mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}\gamma_5) = 0,\quad n = 0,1,\dots.$$

c) Reduce the number of Dirac matrices in the following contractions:

$$\gamma^\alpha\gamma_\alpha,\qquad \gamma^\alpha\gamma^\mu\gamma_\alpha,\qquad \gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha.$$

Please turn over!

Exercise 9.3 *Lorentz covariants from Dirac spinors* (1 point)

a) Prove the following relations:

$$S(\Lambda)^{-1}\,\gamma^\mu\,S(\Lambda) = \Lambda^\mu{}_\nu\,\gamma^\nu, \qquad S(\Lambda)^\dagger\gamma_0 = \gamma_0 S(\Lambda)^{-1}.$$

Use this to show that the quantities

$$\begin{aligned} s(x) &= \bar{\psi}(x)\psi(x) = \text{scalar}, \\ p(x) &= \bar{\psi}(x)\gamma_5\psi(x) = \text{pseudo-scalar}, \\ j^\mu(x) &= \bar{\psi}(x)\gamma^\mu\psi(x) = \text{vector}, \\ j_5^\mu(x) &= \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) = \text{pseudo-vector} \end{aligned}$$

transform under proper, orthochronous Lorentz transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ as indicated, if the Dirac spinor $\psi(x)$ transforms according to $\psi(x) \to \psi'(x') = S(\Lambda)\psi(x)$.

b) Determine the transformation properties of the quantities defined in a) under the parity operation P, where

$$(x'^\mu) = (x^0, -\vec{x}) = (\Lambda_P)^\mu{}_\nu x^\nu, \qquad S(\Lambda_P) = \gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

in the chiral representation of the Dirac matrices.