Exercises to Relativistic Quantum Field Theory — Sheet 7

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Exercise 7.1 *Wick theorem for bosonic fields* (2 points)

The purpose of this exercise is to prove Wick's theorem for bosonic, real field operators $\phi_i \equiv \phi_i(x_i)$, which states that

$$
T\{\phi_1 \cdots \phi_n\} \;=\; :\!\phi_1 \cdots \phi_n\!: \;+\; \sum_{\text{pairs } ij} :\!\phi_1 \cdots \overset{\frown}{\phi_i \cdots \phi_j} \cdots \phi_n\!:
$$

$$
+\; \sum_{\text{double pairs } ij,kl} :\!\phi_1 \cdots \phi_i \cdots \phi_k \cdots \phi_j \cdots \phi_l \cdots \phi_n\!: \;+\; \dots
$$

with the contractions representing propagators, $\overset{\frown}{\phi_i \phi_j} = \langle 0|T\{\phi_i \phi_j\}|0\rangle$. For $n = 2$ the theorem was already proved in Exercise 6.2. We organise the general proof in two steps. Without loss of generality we can assume that $t_n = x_n^0$ is the smallest time variable, i.e. $T\{\phi_1 \cdots \phi_n\} = T\{\phi_1 \cdots \phi_{n-1}\}\phi_n$.

a) First prove the lemma

$$
:\!\phi_1 \cdots \phi_{n-1}\!: \phi_n = :\!\phi_1 \cdots \phi_n\!: \;+\; \sum_{k=1}^{n-1} :\!\phi_1 \cdots \overset{\frown}{\phi_k \cdots \phi_n}\!: .
$$

To this end, split ϕ_n according to $\phi_n = \phi_n^+ + \phi_n^-$ into their positive and negative frequency parts $\phi_n^\pm$, i.e. ϕ_n^+ involves only annihilation operators and ϕ_n^- only creation operators. The contribution owing to ϕ_n^+ is trivially obtained. For ϕ_n^-, proceed via induction in n.

b) Prove Wick's theorem via induction in n using the result of a).

Exercise 7.2 *S-operator for two interacting scalar fields* (1 point)

Consider a theory of a complex scalar field ϕ (particle ϕ and antiparticle $\bar{\phi}$) and a real scalar field Φ (particle Φ) with the Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2 + (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2\phi^\dagger\phi + \mathcal{L}_{\text{int}},$$

where $\mathcal{L}_{\text{int}} = \lambda \phi^\dagger \phi\, \Phi$. Expand the S-operator,

$$S = T \exp\left(i \int \mathrm{d}^4 x\, \mathcal{L}_{\text{int}}(x) \right),$$

up to order λ^2 and use Wick's theorem to express the result in terms of propagators and normal-ordered products of fields. Note that the λ^n contribution can be written in the form

$$\frac{1}{n!} \int \mathrm{d}^4 x_1 \ldots d^4 x_n : \ldots : .$$

Represent the result diagrammatically using the following notation:

- External lines:

$$\phi^\dagger(x) = \overset{x}{\longleftarrow\!\bullet}, \qquad \phi(x) = \overset{x}{\longrightarrow\!\bullet}, \qquad \Phi(x) = \overset{x}{\text{-------}\bullet}$$

- Internal lines:

$$\overline{\phi(x_1)\phi^\dagger}(x_2) = \overset{x_1 \quad x_2}{\bullet\!\longleftarrow\!\bullet}, \qquad \overline{\Phi(x_1)\Phi}(x_2) = \overset{x_1 \quad x_2}{\bullet\text{------}\bullet}$$

- Vertices:

$$i\lambda = $$