

## Exercises to Relativistic Quantum Field Theory — Sheet 2

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### Exercise 2.1 *Classical, relativistic, free point particle* (2.5 points)

We consider a classical, free point particle in special relativity and aim at describing it within the Lagrangian and Hamiltonian formalisms. The particle moves through space on some trajectory  $\vec{x}(t)$  in an inertial frame  $\Sigma$ . We demand that the motion is derived from Hamilton's action principle and that the action  $S$  corresponding to the particle is Lorentz invariant. The only Lorentz-invariant quantity for some motion between the points  $\vec{a}_1 = \vec{x}(t_1)$  and  $\vec{a}_2 = \vec{x}(t_2)$  and fixed times  $t_1$  and  $t_2$  is given by the integral  $\int_{a_1}^{a_2} ds$  with  $a_1 = (ct_1, \vec{a}_1)$ ,  $a_2 = (ct_2, \vec{a}_2)$  and the Lorentz-invariant line element  $ds$  defined by  $ds^2 = c^2 dt^2 - d\vec{x}^2$ . We restrict ourselves to the case where  $ds^2 > 0$ , i.e.  $ds$  is time-like. (In this exercise it is instructive to keep  $c \neq 1$ .)

- a) The above considerations motivate the ansatz

$$S = k \int_{a_1}^{a_2} ds$$

for the action, with  $k$  some preliminary constant. What is the Lagrangian function  $L$  corresponding to  $S$ ? Determine the value of  $k$  by comparing  $L$  to the non-relativistic Lagrangian function  $L_{\text{non-relat}} = \frac{1}{2}m\dot{\vec{x}}^2$ .

[*Result:*  $L = -mc^2/\gamma$  with  $\gamma = 1/\sqrt{1 - v^2/c^2}$ ,  $v = |\dot{\vec{x}}|$ .]

- b) Determine the relativistic three-momentum  $\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}}$  and the Lagrangian equations of motion.
- c) Derive the Hamilton function

$$H(\vec{x}, \vec{p}) = \dot{\vec{x}} \cdot \vec{p} - L(\vec{x}, \dot{\vec{x}}).$$

- d) The four-velocity  $u^\mu$  of the particle is defined by  $(u^\mu) = (\frac{dx^\mu}{ds}) = (\gamma c, \gamma \dot{\vec{x}})$ . Check that the spatial components of the momentum four-vector, defined as  $p^\mu = m u^\mu$ , indeed coincide with  $\vec{p}$  as defined above. What is the relation between the time-like component  $p^0$  and the Hamilton function  $H$ ?
- e) Calculate the invariant square  $p^2$  of  $p^\mu$ . What is the meaning and non-relativistic limit of the resulting relation?

*Please turn over!*

**Exercise 2.2** *Classical, relativistic four-momentum conservation* (1 point)

Consider a system of  $N$  classical point particles, which may interact with each other, but not with any external particles or fields. The Lagrangian function  $L(\vec{x}_i, \dot{\vec{x}}_i, t)$  is, thus, invariant under space and time translations,

$$L(\vec{x}_i, \dot{\vec{x}}_i, t) = L(\vec{x}_i + \vec{a}, \dot{\vec{x}}_i, t + a^0),$$

where  $a^\mu$  is an arbitrary constant four-vector describing the translation.

- a) Derive the conservation of the sum of relativistic three-momenta  $\vec{p}_i = \frac{\partial L}{\partial \dot{\vec{x}}_i}$  from the relation  $\frac{\partial L}{\partial \vec{a}} = 0$ .
- b) Define the Hamilton function  $H$  as the total relativistic energy of the system and show that it is conserved owing to the relation  $\frac{\partial L}{\partial a^0} = 0$ .

**Exercise 2.3** *Mandelstam variables for  $2 \rightarrow 2$  particle reactions* (1.5 points)

We consider the particle reaction  $A(p_1) + B(p_2) \rightarrow C(p_3) + D(p_4)$  with the four-momenta  $p_i$  and  $p_i^2 = m_i^2$ , where  $m_i$  are the particle masses. The Mandelstam variables  $s, t, u$  are defined by  $s = (p_1 + p_2)^2$ ,  $t = (p_1 - p_3)^2$ ,  $u = (p_1 - p_4)^2$ .

- a) Show that  $s + t + u = \sum_{i=1}^4 m_i^2$ .
- b) In the centre-of-mass frame  $\Sigma$  the momenta are given by

$$(p_{1,2}^\mu) = (E_{1,2}, 0, 0, \pm q), \quad (p_{3,4}^\mu) = (E_{3,4}, \pm k \cos \phi \sin \theta, \pm k \sin \phi \sin \theta, \pm k \cos \theta).$$

Calculate  $s, t, u$  in the system  $\Sigma$ . What is the meaning of the variable  $s$ ? Express  $t$  and  $u$  in terms of  $s$ , the angles  $\theta, \phi$ , and the masses  $m_i$ .

- c) Consider the special case  $m_1 = m_3 = 0$ ,  $m_2 = m_4 = m$  in the rest frame  $\Sigma'$  of particle  $B$  (e.g. Compton scattering at an electron at rest). Derive the relation between  $E'_3$  and  $\theta'_3$  for given energy  $E'_1$  upon evaluating  $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$  in two different ways and using energy conservation.