Exercises to Relativistic Quantum Field Theory — Sheet 10

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Exercise 10.1 *Solutions of the free Dirac equation* (1 point)

Consider the solutions of the free Dirac equation in momentum space, $u_\sigma(k)$ and $v_\sigma(k)$ with $\sigma = 1, 2$, which are normalized according to

$$\bar{u}_\sigma(k) u_{\sigma'}(k) = -\bar{v}_\sigma(k) v_{\sigma'}(k) = 2m\, \delta_{\sigma\sigma'}, \qquad \bar{u}_\sigma(k) v_{\sigma'}(k) = \bar{v}_\sigma(k) u_{\sigma'}(k) = 0.$$

a) Prove the completeness relation

$$\sum_{\sigma=1,2} [u_\sigma(k) \otimes \bar{u}_\sigma(k) - v_\sigma(k) \otimes \bar{v}_\sigma(k)] = 2m\, \mathbb{1}.$$

b) Express the matrices

$$\Lambda_+(k) = \frac{1}{2m} \sum_{\sigma=1,2} u_\sigma(k) \otimes \bar{u}_\sigma(k), \qquad \Lambda_-(k) = -\frac{1}{2m} \sum_{\sigma=1,2} v_\sigma(k) \otimes \bar{v}_\sigma(k)$$

in terms of m, $\not{k}$, and the unit matrix $\mathbb{1}$. Argue that $\Lambda_\pm(k)$ are orthogonal projectors onto the subspaces of positive and negative energies, respectively.

c) Show that

$$\bar{u}_\sigma(k) \gamma_\mu u_{\sigma'}(k) = \bar{v}_\sigma(k) \gamma_\mu v_{\sigma'}(k) = 2k_\mu\, \delta_{\sigma\sigma'}$$

by evaluating $\bar{u}_\sigma(k)\{\gamma_\mu, \not{k}\} u_{\sigma'}(k)$ in two different ways.

d) Similarly show that

$$v_\sigma(\tilde{k})^\dagger u_{\sigma'}(k) = u_\sigma(\tilde{k})^\dagger v_{\sigma'}(k) = 0,$$

using $\gamma_0 \not{k} = \tilde{\not{k}} \gamma_0$, where $\tilde{k}^\mu = (k_0, -\vec{k})$ for $k^\mu = (k_0, \vec{k})$.

Exercise 10.2 *Polarisation sums for Dirac spinors* (0.5 points)

As in Exercise 10.1, the quantities $u_\sigma(p)$ and $v_\tau(P)$ with $\sigma, \tau = 1, 2$ denote the solutions of the free Dirac equation in momentum space, where $p^2 = m^2$, $P^2 = M^2$. Show that the following polarisation sum $\sum_{\sigma,\tau}$ can be written as a trace in Dirac space according to

$$\sum_{\sigma,\tau} \Big(\bar{u}_\sigma(p) \Gamma v_\tau(P)\Big)^* \Big(\bar{u}_\sigma(p) \Gamma v_\tau(P)\Big) = \mathrm{Tr}\Big[(\not{P} - M)\tilde{\Gamma}(\not{p} + m)\Gamma\Big].$$

Here Γ is an arbitrary 4×4 matrix and $\tilde{\Gamma} = \gamma_0 \Gamma^\dagger \gamma_0$.

Please turn over!

Exercise 10.3 *Field operator of the free Dirac fermion* (1 point)

Consider the following plane-wave expansion of the field operator $\psi(x)$ of the free Dirac fermion,

$$\psi(x) = \int \mathrm{d}\tilde{p} \sum_\sigma \left(\mathrm{e}^{-ipx} u_\sigma(p) a_\sigma(\vec{p}) + \mathrm{e}^{+ipx} v_\sigma(p) b_\sigma^\dagger(\vec{p}) \right),$$

where $a_\sigma^{(\dagger)}(\vec{p})$ and $b_\sigma^{(\dagger)}(\vec{p})$ denote the annihilation (creation) operators of the particle and antiparticle states, respectively, which obey the anticommutation relations

$$\{a_\sigma(\vec{p}), a_\tau^\dagger(\vec{k})\} = \{b_\sigma(\vec{p}), b_\tau^\dagger(\vec{k})\} = 2p_0(2\pi)^3 \delta_{\sigma\tau}\delta(\vec{p}-\vec{k}), \qquad \{a_\sigma(\vec{p}), a_\tau(\vec{k})\} = \cdots = 0.$$

a) Calculate the operator

$$\hat{Q} = Qe \int \mathrm{d}^3x \; :\bar{\psi}(x)\gamma_0\psi(x):$$

of electric charge in terms of an integral in momentum space.

b) Derive the commutators $[\hat{Q}, a_\sigma^{(\dagger)}(\vec{p})]$ and $[\hat{Q}, b_\sigma^{(\dagger)}(\vec{p})]$ and interpret the result.

Exercise 10.4 *Free Dirac propagator* (1 bonus point)

The free Dirac propagator is explicitly given by

$$S_F(x,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \, \mathrm{e}^{-ik(x-y)} \, \frac{\not{k}+m}{k^2-m^2+i\epsilon}.$$

a) Upon carrying out the k_0-integration, show that $S_F(x,y)$ can be written as

$$\begin{aligned} S_F(x,y) = &- i\theta(x_0-y_0) \int \mathrm{d}\tilde{k} \, \mathrm{e}^{-ik(x-y)}(m+\not{k}) \\ &- i\theta(y_0-x_0) \int \mathrm{d}\tilde{k} \, \mathrm{e}^{+ik(x-y)}(m-\not{k}). \end{aligned}$$

b) We denote the solutions of the free Dirac equation with positive and negative energies $\psi^{(+)}(x)$ and $\psi^{(-)}(x)$, respectively. Show that

$$\begin{aligned} \theta(x_0-y_0)\psi^{(+)}(x) &= \quad i \int \mathrm{d}^3y \, S_F(x,y)\gamma_0\psi^{(+)}(y), \\ \theta(y_0-x_0)\psi^{(-)}(x) &= - i \int \mathrm{d}^3y \, S_F(x,y)\gamma_0\psi^{(-)}(y). \end{aligned}$$