
Exercises to Relativistic Quantum Field Theory — Sheet 2

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Exercise 2.1 *Classical, relativistic, free point particle* (2 points)

We consider a classical, free point particle in special relativity and aim at describing it within the Lagrangian and Hamiltonian formalisms. The particle moves through space on some trajectory $\vec{x}(t)$ in an inertial frame Σ . We demand that the motion is derived from Hamilton's action principle and that the action S corresponding to the particle is Lorentz invariant. The only Lorentz-invariant quantity for some motion between the points $\vec{a}_1 = \vec{x}(t_1)$ and $\vec{a}_2 = \vec{x}(t_2)$ and fixed times t_1 and t_2 is given by the integral $\int_{\vec{a}_1}^{\vec{a}_2} ds$ with the Lorentz-invariant line element ds defined by $ds^2 = c^2 dt^2 - d\vec{x}^2$. (In this exercise it is instructive to keep $c \neq 1$.)

- a) The above considerations motivate the ansatz

$$S = k \int_{(ct_1, \vec{a}_1)}^{(ct_2, \vec{a}_2)} ds$$

for the action, with k some preliminary constant. What is the Lagrangian function L corresponding to S ? Determine the value of k by comparing L with the non-relativistic Lagrangian function $L_{\text{non-relat}} = \frac{1}{2}m\dot{\vec{x}}^2$.

[Result: $L = -mc^2/\gamma$ with $\gamma = 1/\sqrt{1 - v^2/c^2}$, $v = |\dot{\vec{x}}|$.]

- b) Determine the relativistic three-momentum $\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}}$ and the Lagrangian equations of motion.
- c) Derive the Hamilton function

$$H(x^k, p^k) = \sum_{\ell=1}^3 \dot{x}^\ell p^\ell - L(x^k, \dot{x}^k).$$

- d) The four-velocity u^μ of the particle is defined by $u^\mu = (\gamma c, \gamma \dot{\vec{x}})$. Check that the space-like components of the momentum four-vector, defined as $p^\mu = m u^\mu$, indeed coincide with \vec{p} as defined above. What is the relation between the time-like component p^0 and the Hamilton function H ?
- e) Calculate the invariant square p^2 of p^μ . What is the meaning and non-relativistic limit of the resulting relation?

Please turn over!

Exercise 2.2 *Classical, relativistic four-momentum conservation* (1 point)

Consider a system of N classical point particles, which may interact with each other, but not with any external particles or fields. The Lagrangian function $L(\vec{x}_i, \dot{\vec{x}}_i, t)$ is, thus, invariant under space and time translations,

$$L(\vec{x}_i, \dot{\vec{x}}_i, t) = L(\vec{x}_i + \vec{a}, \dot{\vec{x}}_i, t + a^0),$$

where a^μ is an arbitrary constant four-vector describing the translation.

- a) Derive the conservation of the sum of relativistic three-momenta $\vec{p}_i = \frac{\partial L}{\partial \dot{\vec{x}}_i}$ from the relation $\frac{\partial L}{\partial \vec{a}} = 0$.
- b) Define the Hamilton function H as the total relativistic energy of the system and show that it is conserved owing to the relation $\frac{\partial L}{\partial a^0} = 0$.

Exercise 2.3 *Mandelstam variables for $2 \rightarrow 2$ particle reactions* (2 points)

We consider the particle reaction $A(p_1) + B(p_2) \rightarrow C(p_3) + D(p_4)$ with the four-momenta p_i and $p_i^2 = m_i^2$, where m_i are the particle masses. The Mandelstam variables s, t, u are defined by $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$.

- a) Show that $s + t + u = \sum_{i=1}^4 m_i^2$.
- b) In the centre-of-mass frame Σ the momenta are given by

$$p_{1,2} = (E_{1,2}, 0, 0, \pm q), \quad p_{3,4} = (E_{3,4}, \pm k \cos \phi \sin \theta, \pm k \sin \phi \sin \theta, \pm k \cos \theta).$$

Which constraints result from four-momentum conservation?

- c) Calculate s, t, u in the system Σ . What is the meaning of the variable s ? Express t and u in terms of s , the angles θ, ϕ , and the masses m_i .
- d) Consider the special case $m_1 = m_3 = 0$, $m_2 = m_4 = m$ in the rest frame Σ' of particle B (e.g. Compton scattering at an electron at rest). Derive the relation between E'_3 and θ'_3 for given energy E'_1 upon evaluating $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$ in two different ways and using energy conservation.