Exercises to Group Theory for Physicists — Sheet 7
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Exercise 7.1 *Tensors of SU(3) and SO(3)* (2 points)

a) Use the invariant symbols δ^i_j, ϵ^{ijk}, ϵ_{ijk} of SU(3) and symmetry properties to find the irreducible contributions of a rank-3 tensor T^{ijk} of SU(3) with 3 upper and no lower indices. What is the corresponding Clebsch-Gordan series?

   Note: It is sufficient to give the irreducible contributions as tensors of lower rank, where appropriate.

b) How does the result change if we regard a tensor T^{ijk} of SO(3) instead? Explain (without performing the full calculation) what the contributions to the corresponding Clebsch-Gordan series are.

Exercise 7.2 *Representations of SO(4)* (3 points)

In the lecture we showed that SO(4) is locally isomorphic to SU(2)$\times$SU(2). Furthermore, SU(2) $\times$ SU(2) is a universal cover of SO(4).

a) Use your knowledge about SU(2) to construct representations of SO(4) from two irreducible representations of SU(2). What are the dimensions of those SO(4) representations?

b) Show that these representations are irreducible.

c) Find a factor group of SU(2) $\times$ SU(2) that is isomorphic to SO(4).

Exercise 7.3 *The energy spectrum of the hydrogen atom* (4 points)

The Hamiltonian H and the energy eigenvalues E_n corresponding to the eigenstates $|n,l,m\rangle$ of H, $\vec{L}^2$, L_3 ($\vec{L}$ is the angular momentum operator), of the hydrogen atom are given by

$$\begin{aligned} H|n,l,m\rangle &= E_n|n,l,m\rangle, & H &= \frac{p^2}{2m} - \frac{k}{r}, \quad E_n = -\frac{R_\infty}{n^2}, \quad n \in \mathbb{N}_1, \\ \vec{L}^2|n,l,m\rangle &= \hbar^2 l(l+1)|n,l,m\rangle, & l &\in \mathbb{N}_0 \text{ with } 0 \le l \le n-1, \\ L_3|n,l,m\rangle &= \hbar m|n,l,m\rangle, & m &\in \mathbb{Z} \text{ with } -l \le m \le l, \end{aligned} \tag{1}$$

with $k > 0$ and $R_\infty = \frac{mk^2}{2\hbar^2}$. From classical mechanics we know that there exists a conserved vector

$$\vec{A}_{\text{cl.}} = \frac{1}{m}\vec{L} \times \vec{p} + k\frac{\vec{x}}{r} \tag{2}$$

called Laplace-Runge-Lenz vector that points from the centre to the perihelion of the trajectory (this implies that fact that the bound trajectories in a r^{-1} potential are closed). In quantum mechanics we define the Laplace-Runge-Lenz operator

$$\vec{A} = \frac{1}{m}(\vec{L} \times \vec{p} - \vec{p} \times \vec{L}) + k\frac{\vec{x}}{r} \tag{3}$$

by symmetrisation so that the components $A_i = A_i^\dagger$, $i = 1, 2, 3$, are hermitian.

a) It can be shown that $\vec{A}$ fulfils the commutator relations

$$[H, \vec{A}] = 0, \qquad [L_i, A_j] = \mathrm{i}\epsilon_{ijk} A_k, \qquad [A_i, A_j] = \mathrm{i}\epsilon_{ijk}\left(-\frac{2}{m} H L_k\right). \tag{4}$$

What is the symmetry generated by $\vec{L}$ and $\vec{A}$ and what is the expected degeneracy of the energy eigenvalues? Compare this to the observed degeneracy and the degeneracy induced by SO(3).

Hint: Introduce the operators $\vec{T}_\pm = \frac{1}{2}(\vec{L} \pm \vec{M})$ with $\vec{M} = \sqrt{-\frac{m}{2H}}\vec{A}$.

b) Calculate $\vec{L} \cdot \vec{A}$ for the operators $\vec{L}$ and $\vec{A}$.

c) Express the relation from b) in terms of $\vec{T}_\pm$. What does this imply for the degrees of degeneracy?

d) Express $\frac{1}{2}(\vec{L}^2 + \vec{M}^2)$ in terms of $\vec{T}_\pm$ and use

$$\vec{A}^2 = \frac{2H}{m}(\vec{L}^2 + \hbar^2) + k^2 \tag{5}$$

to derive a formula for the energy eigenvalues.

Comment: Although the calculations are a bit tedious, we recommend to prove the commutator relations (4) and the formula (5) explicitly.