Exercises to Relativistic Quantum Field Theory — Sheet 8
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Exercise 8.1 *S-operator for two interacting scalar fields (cont'd)* (1 point)

Consider again the field theory of a complex scalar field ϕ (particle ϕ and antiparticle $\bar{\phi}$) and a real scalar field Φ (particle Φ) from Exercise 7.2 with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2 + (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2\phi^\dagger\phi + \mathcal{L}_{\text{int}},$$

where $\mathcal{L}_{\text{int}} = \lambda\phi^\dagger\phi\,\Phi$, and make use of the perturbative expansion worked out there.

a) Calculate the S-matrix element $S_{fi} = \langle f|S|i\rangle$ in lowest order between the initial state $|i\rangle = a_\Phi^\dagger(k)|0\rangle$ and the final state $|f\rangle = a_\phi^\dagger(p_1)\, b_\phi^\dagger(p_2)|0\rangle$, where $a_\Phi^\dagger(q)$, $a_\phi^\dagger(q)$, $b_\phi^\dagger(q)$ are the creation operators of the particles Φ, ϕ, and $\bar{\phi}$.

b) Assuming $M > 2m$, calculate the lowest-order decay width

$$\Gamma_{\Phi\to\phi\bar{\phi}} = \frac{1}{2M}\int \mathrm{d}\Phi_2\, |\mathcal{M}_{fi}|^2$$

for the decay $\Phi \to \phi\bar{\phi}$, where Φ_2 is the 2-particle phase space of the final state (see Exercise 5.2) and the transition matrix element $\mathcal{M}_{fi}$ is related to S_{fi} by

$$S_{fi} = (2\pi)^4\delta(k - p_1 - p_2)\, i\mathcal{M}_{fi}.$$

Exercise 8.2 *Fundamental representations of the Lorentz group* (1 point)

The general form of Lorentz transformations in the two fundamental representations of the Lorentz group is given by

$$\Lambda_{\mathrm{R}} = \exp\left(-\frac{\mathrm{i}}{2}\sum_{k=1}^{3}(\phi_k + i\nu_k)\sigma^k\right), \qquad \Lambda_{\mathrm{L}} = \exp\left(-\frac{\mathrm{i}}{2}\sum_{k=1}^{3}(\phi_k - i\nu_k)\sigma^k\right)$$

with the real group parameters ϕ_k, ν_k and the Pauli matrices σ^k.

a) Show that $\Lambda_{\mathrm{R}}^\dagger = \Lambda_{\mathrm{L}}^{-1}$ and $\Lambda_{\mathrm{L}}^\dagger = \Lambda_{\mathrm{R}}^{-1}$.

b) Show that $\det(\Lambda_{\mathrm{R}}) = \det(\Lambda_{\mathrm{L}}) = 1$ using $\det(\exp\{A\}) = \exp\{\mathrm{Tr}(A)\}$ for a matrix A.

c) Which transformations are characterized by $\Lambda_{\mathrm{R/L}}^\dagger = \Lambda_{\mathrm{R/L}}$, which by $\Lambda_{\mathrm{R/L}}^\dagger = \Lambda_{\mathrm{R/L}}^{-1}$?

d) Derive Λ_{R} and Λ_{L} for a pure boost in the direction $\vec{e} = \begin{pmatrix}\cos\varphi\sin\theta\\ \sin\varphi\sin\theta\\ \cos\theta\end{pmatrix}$ with $\vec{\nu} = \nu\vec{e}$, $\vec{\phi} = 0$ and for a pure rotation around the axis $\vec{e}$ with $\vec{\phi} = \phi\vec{e}$, $\vec{\nu} = 0$.

Please turn over!

Exercise 8.3 *Connection between $\Lambda_{\rm R}$, $\Lambda_{\rm L}$, and $\Lambda^\mu{}_\nu$* (1 point)

The general matrix representing a Lorentz transformation of a four-vector is given by

$$\Lambda^\mu{}_\nu = \exp\left(-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right)^\mu{}_\nu, \qquad (M^{\alpha\beta})^\mu{}_\nu = i(g^{\alpha\mu}g^\beta{}_\nu - g^{\beta\mu}g^\alpha{}_\nu)$$

with the antisymmetric parameters $\omega_{jk} = \epsilon_{jkl}\phi_l$ and $\omega_{0j} = -\omega_{j0} = \nu_j$. The connection between $\Lambda_{\rm R}$, $\Lambda_{\rm L}$ (see Exercise 8.1) and $\Lambda^\mu{}_\nu$ is

$$\Lambda_{\rm R}^\dagger\sigma^\mu\Lambda_{\rm R} = \Lambda^\mu{}_\nu\sigma^\nu, \qquad \Lambda_{\rm L}^\dagger\bar{\sigma}^\mu\Lambda_{\rm L} = \Lambda^\mu{}_\nu\bar{\sigma}^\nu,$$

where $\sigma^\mu = (\mathbb{1}, \sigma^1, \sigma^2, \sigma^3)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^1, -\sigma^2, -\sigma^3)$. Verify these relations for infinitesimal transformations with the parameters $\delta\phi_k$, $\delta\nu_k$.