
Exercises to Relativistic Quantum Field Theory — Sheet 5

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Exercise 5.1 *Momentum of the quantised free scalar field* (1 point)

The energy P^0 and momentum \vec{P} of a free real scalar field $\phi(x)$ are determined by the energy-momentum tensor via

$$P^0 = \int d^3y \frac{1}{2} [\pi(y)^2 + (\nabla\phi(y))^2 + m^2\phi(y)^2], \quad \vec{P} = - \int d^3y \pi(y)(\nabla\phi(y)),$$

where $\pi(x) = \dot{\phi}(x)$ is the canonical conjugate operator to the field operator $\phi(x)$.

- a) Using the canonical commutator relations, show that

$$[\phi(x), P^\mu] = i\partial^\mu\phi(x).$$

- b) Derive the following identity for a translation by a constant 4-vector a^μ :

$$\exp(ia_\mu P^\mu) \phi(x) \exp(-ia_\nu P^\nu) = \phi(x + a).$$

Exercise 5.2 *2-particle phase space* (1 point)

We consider two particles with masses $m_{1,2}$ and four-momenta $p_{1,2}$ ($p_{1,2}^2 = m_{1,2}^2$). The total momentum is, thus, given by $k = p_1 + p_2$. The integral over the 2-particle phase space is defined as

$$\int d\Phi_2 = \int \frac{d^3p_1}{(2\pi)^3 2p_1^0} \int \frac{d^3p_2}{(2\pi)^3 2p_2^0} (2\pi)^4 \delta^{(4)}(k - p_1 - p_2) \Big|_{p_{1,2}^0 = \sqrt{m_{1,2}^2 + \vec{p}_{1,2}^2}}.$$

- a) Consider the decay of a particle of mass M and momentum k ($k^2 = M^2$) into two particles with momenta $p_{1,2}$. Show that the phase-space integral can be evaluated in the centre-of-mass frame as

$$\int d\Phi_2 = \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{8M^2} \theta(M - m_1 - m_2) \int d\Omega_1,$$

where Ω_1 is the solid angle of particle 1, and $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$.

- b) What is the phase-space integral of a $2 \rightarrow 2$ particle scattering reaction with incoming momenta $k_{1,2}$. What is the counterpart of M in this reaction?

Exercise 5.3 *Charge operator of the free complex scalar field* (0.5 points)

Consider the field operator $\phi(x)$ of the free complex Klein-Gordon field describing a spin-0 boson of mass m and electric charge q . The plane-wave expansion of $\phi(x)$ is given by

$$\phi(x) = \int d\tilde{p} \left(a(\vec{p}) e^{-ipx} + b^\dagger(\vec{p}) e^{ipx} \right),$$

where $a(\vec{p})$, $a^\dagger(\vec{p})$ are the annihilation and creation operators for the particle, respectively, and likewise $b(\vec{p})$, $b^\dagger(\vec{p})$ for the corresponding antiparticle. Express the charge operator

$$Q = \int d^3\vec{x} i q : \left(\phi^\dagger (\partial_0 \phi) - (\partial_0 \phi)^\dagger \phi \right) :$$

in terms of the annihilation and creation operators of the momentum eigenstates. The normal ordered form $:\mathcal{O}:$ of an operator \mathcal{O} is defined in such a way that all annihilation operators a_i are to the right of all creation operators a_j^\dagger (e.g. $:a_i a_j^\dagger: = :a_j^\dagger a_i: = a_j^\dagger a_i$).