Exercises to Relativistic Quantum Field Theory — Sheet 3

— Prof. S. Dittmaier, Universität Freiburg, SS16 —

Exercise 3.1 *Green's function of Schrödinger's equation* (1 point)

The *Green function* G of Schrödinger's equation of a particle of mass m in the potential V is defined via the condition

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2m}\nabla_x^2 - V(\vec{x})\right) G(t,\vec{x};t',\vec{x'}) = \delta(t-t')\delta^{(3)}(\vec{x}-\vec{x'}).$$

a) Determine the Fourier transform of the Green function G_0 of the *free* Schrödinger equation with $V(\vec{x}) = 0$, i.e. write G_0 in the form

$$G_0(t,\vec{x};t',\vec{x'}) = \frac{1}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}^3p\, e^{-i\omega(t-t')} e^{i\vec{p}\cdot(\vec{x}-\vec{x'})}\tilde{G}_0(\omega,\vec{p})$$

and determine the function $\tilde{G}_0(\omega,\vec{p})$. Why can the free Green function only depend on $t-t'$ and $\vec{x}-\vec{x'}$?

b) Show that the Green function of the full Schrödinger equation with $V \neq 0$ fulfills the equation

$$G(t,\vec{x};t',\vec{x'}) = G_0(t,\vec{x};t',\vec{x'}) + \int \mathrm{d}\tilde{t}\, \mathrm{d}^3\tilde{x}\; G_0(t,\vec{x};\tilde{t},\tilde{\vec{x}})V(\tilde{\vec{x}})G(\tilde{t},\tilde{\vec{x}};t',\vec{x'}).$$

Exercise 3.2 *Hamiltonian of the classical, free Klein-Gordon field* (1.5 points)

Consider the Lagrangian $\mathcal{L} = (\partial\phi)(\partial\phi)^* - m^2\phi\phi^*$ of the free, complex Klein-Gordon field ϕ.

a) Derive the corresponding Hamiltonian $\mathcal{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}$, with $\pi = \dfrac{\partial\mathcal{L}}{\partial\dot{\phi}}$ denoting the canonical conjugate field to ϕ.

b) Using the plane-wave solution $\phi = \int \mathrm{d}\tilde{p}\,\left[a(\vec{p})\mathrm{e}^{-ipx} + b^*(\vec{p})\mathrm{e}^{+ipx}\right]$ with some (square-integrable) arbitrary complex functions $a(\vec{p})$, $b(\vec{p})$, show that

$$H = \int \mathrm{d}^3x\,\mathcal{H} = \int \mathrm{d}\tilde{p}\,p_0\,\left[|a(\vec{p})|^2 + |b(\vec{p})|^2\right],$$

where the momentum-space integral is defined by $\int \mathrm{d}\tilde{p} \equiv \int \left.\dfrac{\mathrm{d}^3p}{(2\pi)^3 2p_0}\right|_{p_0=\sqrt{\vec{p}^2+m^2}}$.

Please turn over!

c) Employing the (time-independent) scalar product

$$(\phi, \chi) \equiv i \int \mathrm{d}^3x \; [\phi(x)^* (\partial_0 \chi(x)) - \chi(x)(\partial_0 \phi(x)^*)]$$

of two free Klein-Gordon fields ϕ, χ, is it possible to interpret H as the expectation value of the energy operator $P_0 = i\partial_0$ of the field modes, i.e. as $(\phi, P_0\phi)$?

Exercise 3.3 *Gauge invariance and minimal substitution* (1 point)

a) The electric and magnetic fields $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi$ and $\vec{B} = \nabla \times \vec{A}$ are invariant under the *gauge transformation* $\phi \to \phi + \frac{\partial \omega}{\partial t}$, $\vec{A} \to \vec{A} - \nabla\omega$ of the scalar and vector potentials, where $\omega(t, \vec{x})$ is an arbitrary function of space and time.

Show that Schrödinger's equation of a spinless particle with charge q in the electromagnetic field,

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2m}\left(\nabla_x - iq\vec{A}(t,\vec{x})\right)^2 - q\phi(t,\vec{x}) \right) \psi(t,\vec{x}) = 0,$$

is invariant under gauge transformations if the following transformation of the wave function ψ is applied simultaneously:

$$\psi(t,\vec{x}) \to e^{-iq\omega(t,\vec{x})}\psi(t,\vec{x}) \,.$$

Comment: The introduction of the interaction with the electromagnetic field by the replacements $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + iq\phi$ and $\nabla \to \nabla - iq\vec{A}$ is called *minimal substitution.*

b) Carrying out the minimal substitution in the free Klein-Gordon equation of a scalar field with charge q we obtain the Klein-Gordon equation with the electromagnetic interaction:

$$\left[(\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) + m^2 \right] \Phi(x) = 0$$

with $A^\mu = (\phi, \vec{A})$. Show in analogy to Schrödinger's equation that this equation is invariant under gauge transformations if the following transformation of the scalar field is applied:

$$\Phi'(x) \to e^{-iq\omega(x)}\Phi(x) \,.$$