Exercise 6.1 *Landau levels of electrons in a magnetic field* (2 points)

Consider an electron (electric charge $q = -e$) in a homogeneous magnetic field, which is aligned along the x_3 axis ($\mathbf{B} = \nabla \times \mathbf{A} = B\mathbf{e_3}$).

a) Generate the Hamilton operator upon applying the "minimal substitution" $\hat{\mathbf{p}} \rightarrow \hat{\mathbf{\Pi}} = \hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{x}})$ to the Hamilton operator of a free particle and subsequently adding the interaction part $\hat{H}_s = -\vec{\mu} \cdot \vec{B}$ of the magnetic field $\vec{B}$ with the magnetic moment $\vec{\mu} = \frac{gq}{2m_\mathrm{e}}\vec{S}$ induced by the spin $\vec{S}$.

b) Reduce the eigenvalue problem of the Hamilton operator to appropriate one-dimensional problems for the spatial motion and show that the eigenvalues have the form

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \hbar\omega\left(n + \frac{1}{2}\right) + g_\mathrm{e}\hbar\omega_\mathrm{L} m_s, \qquad n \in \mathbf{N}_0, \quad m_s = \pm\frac{1}{2},$$

where $\hbar k$ is the continuous eigenvalue of $\hat{p}_3$, m_s corresponds to the spin orientation, and $g_\mathrm{e} = 2.002...$ denotes the g-factor of the positron.

Exercise 6.2 *SU(2) matrices and rotations* (2 points)

SU(2) is the Lie group of dimension 3 consisting of all complex, unitary 2×2 matrices A with $\det A = 1$. A convenient way to parametrize A is in terms of a rotation angle θ ($0 \leq \theta < 2\pi$) and a 3-dim. real unit vector $\vec{e}$ (e.g. parametrized by its polar and azimuthal angles ϑ and φ, respectively), defining the "rotation vector" $\vec{\theta} = \theta\vec{e}$:

$$A(\vec{\theta}) \;=\; \exp\left\{-\frac{\mathrm{i}}{2}\vec{\theta}\cdot\vec{\sigma}\right\} \;=\; \cos\frac{\theta}{2}\,\mathbf{1} - \mathrm{i}(\vec{e}\cdot\vec{\sigma})\sin\frac{\theta}{2}, \tag{1}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the "vector" formed by the Pauli matrices σ_a.

a) Show that all $A \in \mathrm{SU}(2)$ can be expressed in terms of the form (1). Express θ and $\vec{e}$ in terms of the coefficients of A. Which matrices A correspond to a given rotation in 3-dim. space?

b) Associating a matrix $V = \vec{v}\cdot\vec{\sigma}$ with each 3-dim. vector $\vec{v}$, show that the transformation $V \rightarrow V' = AVA^\dagger$ rotates the vector $\vec{v}$ into the vector $\vec{v}\,' = R(\vec{\theta})\vec{v}$ corresponding to the matrix $V' = \vec{v}\,'\cdot\vec{\sigma}$, where $R(\vec{\theta})$ is the general rotation matrix of Exercise 5.2. Show that the different versions of A corresponding to the same rotation in 3-dim. space in fact lead to the same rotated vector $\vec{v}\,'$.

Please turn over!

Exercise 6.3 *Dynamical symmetry of the isotropic 3-dim. harmonic oscillator* (4 points)

In Exercise 2.1 you have decomposed the Hamiltonian $\hat{H}$ of the isotropic 3-dimensional harmonic oscillator into its individual parts corresponding to cartesian coordinates, resulting in

$$\hat{H} = \hbar\omega \sum_{j=1}^{3} \left(a_j^\dagger a_j + \frac{1}{2} \right),$$

where a_j and $a_j^\dagger$ $(j = 1, 2, 3)$ are the usual shift operators for the movement in x_i direction obeying the relations

$$[a_j, a_k] = 0, \qquad [a_j^\dagger, a_k^\dagger] = 0, \qquad [a_j, a_k^\dagger] = \delta_{jk}.$$

a) For which complex matrices U does the replacement $\vec{a} = (a_1, a_2, a_3)^T \to \vec{a}' = U\vec{a}$ represent a symmetry? The set of all U defines a Lie group. What is the dimension of this group? How are the generators X_a of this group characterized?

b) What is the role of the one-dimensional subgroup consisting of pure phase transformations $U = \mathrm{e}^{\mathrm{i}\theta}\mathbf{1}$? What is an appropriate condition on the matrices U of a) to eliminate those phase transformations?

c) What is the relation of the symmetry transformations represented by U defined in a) to rotations in 3-dim. space? What is the relation between the X_a and orbital angular momentum? Identify the "accidental symmetry" that goes beyond pure rotational invariance.

d) The energy eigenstates to the eigenvalue $E_n = \hbar\omega(n + \frac{3}{2})$ with a fixed number $n \in \mathbf{N}_0$ are proportional to the states $a_{j_1}^\dagger \ldots a_{j_n}^\dagger |0\rangle$, which transform under U like components of a symmetric tensors of rank n in three dimensions:

$$a_{j_1}^\dagger \ldots a_{j_n}^\dagger |0\rangle \to a_{j_1}'^\dagger \ldots a_{j_n}'^\dagger |0\rangle = \sum_{k_1,\ldots,k_n} U_{j_1 k_1}^* \ldots U_{j_n k_n}^* a_{k_1}^\dagger \ldots a_{k_n}^\dagger |0\rangle.$$

Deduce the degree of degeneracy of E_n from this consideration.