Exercises on Supersymmetry **Sheet 1**

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Exercise 1 *Fundamental representations of the Lorentz group* (5 points)

The general form of a Lorentz transformation in the two fundamental representations of the Lorentz-group reads

$$\Lambda_{\mathrm{R}} = \exp\left\{-\frac{i}{2}(\phi_k + i\nu_k)\sigma^k\right\}, \qquad \Lambda_{\mathrm{L}} = \exp\left\{-\frac{i}{2}(\phi_k - i\nu_k)\sigma^k\right\}, \tag{1}$$

with real parameters ϕ_k, ν_k, $k = 1, 2, 3$ and the Pauli matrices σ^k.

a) Show that $\Lambda_{\mathrm{R}}^\dagger = \Lambda_{\mathrm{L}}^{-1}$ and $\Lambda_{\mathrm{L}}^\dagger = \Lambda_{\mathrm{R}}^{-1}$.

b) Using the identity $\det(\exp A) = \exp(\operatorname{tr} A)$ for matrices A show that $\det \Lambda_{\mathrm{R}} = \det \Lambda_{\mathrm{L}} = 1$.

c) For which transformations is $\Lambda_{\mathrm{R/L}}^\dagger = \Lambda_{\mathrm{R/L}}$ true and for which $\Lambda_{\mathrm{R/L}}^\dagger = \Lambda_{\mathrm{R/L}}^{-1}$?

d) Calculate Λ_{R} and Λ_{L} for a pure boost ($\boldsymbol{\phi} = 0$) in direction $\boldsymbol{e} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}$ with $\boldsymbol{\nu} = \nu \boldsymbol{e}$, and also for a pure rotation ($\boldsymbol{\nu} = 0$) around $\boldsymbol{e}$ with $\boldsymbol{\phi} = \phi \boldsymbol{e}$.

e) Within the Weyl–van-der-Waerden calculus covariant spinors ξ_a transform with Λ_{R} and contravariant spinors $\bar{\eta}^{\dot{a}}$ with Λ_{L}. What does that imply for the indices (placement, dotting) of the matrices $\Lambda_{\mathrm{R/L}}$ and $\Lambda_{\mathrm{R/L}}^*$? Show that the spinor calculus (raising, lowering, dotting of indices) is consistent with the relation $\epsilon \Lambda_{\mathrm{R}}^* \epsilon^{-1} = \Lambda_{\mathrm{L}}$.

Exercise 2 *Relation between Λ_{R}, Λ_{L}, and $\Lambda^\mu{}_\nu$* (4 points)

For four-vectors the general matrix of a Lorentz transformation is

$$\Lambda^\mu{}_\nu = \exp\left\{-\frac{i}{2}\omega_{\alpha\beta} M^{\alpha\beta}\right\}^\mu{}_\nu, \qquad \left(M^{\alpha\beta}\right)^\mu{}_\nu = i\left(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu\right), \tag{2}$$

with antisymmetric parameters $\omega_{jk} = \epsilon_{jkl}\phi_l$ and $\omega_{0j} = -\omega_{j0} = \nu_j$. Verify the following relations between Λ_{R}, Λ_{L}, and $\Lambda^\mu{}_\nu$,

$$\Lambda_{\mathrm{R}}^\dagger \sigma^\mu \Lambda_{\mathrm{R}} = \Lambda^\mu{}_\nu \sigma^\nu, \qquad \Lambda_{\mathrm{L}}^\dagger \bar{\sigma}^\mu \Lambda_{\mathrm{L}} = \Lambda^\mu{}_\nu \bar{\sigma}^\nu, \tag{3}$$

for infinitesimal parameters $\delta\phi_k$ and $\delta\nu_k$ where $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$ are the four-dimensional Pauli matrices. Accustom yourself to the notation of the spinor calculus.

Exercise 3 *Relation between the Lorentz group and* $\mathrm{SL}(2,\mathbf{C})$ (5 points)

The group $\mathrm{SL}(2,\mathbf{C})$ is the set of all complex 2×2 matrices A with $\det A = 1$. Consider the following mappings of four-vectors x^μ onto 2×2 matrices,

$$X^{\dot{a}b} = x_\mu \sigma^{\mu,\dot{a}b}, \tag{4}$$
$$\bar{X}_{a\dot{b}} = x_\mu \bar{\sigma}^\mu_{a\dot{b}}, \tag{5}$$

where $\sigma^{\mu,\dot{a}b} = \left(\mathbf{1}^{\dot{a}b}, \boldsymbol{\sigma}^{\dot{a}b}\right)$ and $\bar{\sigma}^\mu_{a\dot{b}} = (\mathbf{1}_{a\dot{b}}, -\boldsymbol{\sigma}_{a\dot{b}})$ are the entries of the four-dimensional Pauli-matrices which fulfill the relation $\mathrm{tr}\,(\sigma^\mu\bar{\sigma}^\nu) = \sigma^{\mu,\dot{a}b}\bar{\sigma}^\nu_{b\dot{a}} = 2g^{\mu\nu}$.

a) Show that the relations $x^\mu = \frac{1}{2}\,\mathrm{tr}\,(X\bar{\sigma}^\mu) = \frac{1}{2}\,\mathrm{tr}\left(\bar{X}\sigma^\mu\right)$ are valid and thus the inverse of the mapping given above.

b) What is the interpretation of $\det(X)$ and $\det(\bar{X})$?

c) Given an arbitrary matrix $A \in \mathrm{SL}(2,\mathbf{C})$. Show that the mappings $X \mapsto X' = AXA^\dagger$ and $\bar{X} \mapsto \bar{X}' = A\bar{X}A^\dagger$ define Lorentz transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.
(Hint: Examine the determinants.)

d) What is the relation between the matrix A from c) and the matrices Λ_R and Λ_L of the fundamental representation?
(Hint: Make use of the relations $\Lambda_\mathrm{R}^\dagger\sigma^\mu\Lambda_\mathrm{R} = \Lambda^\mu{}_\nu\sigma^\nu$ and $\Lambda_\mathrm{L}^\dagger\bar{\sigma}^\mu\Lambda_\mathrm{L} = \Lambda^\mu{}_\nu\bar{\sigma}^\nu$.)

e) In c), you have shown that each $A \in \mathrm{SL}(2,\mathbf{C})$ defines a $\Lambda \in L_+^\uparrow$. Assuming that each $\Lambda \in L_+^\uparrow$ can be represented by an element of $\mathrm{SL}(2,\mathbf{C})$, show that exactly two elements of $\mathrm{SL}(2,\mathbf{C})$ correspond to a given Λ.

Group-theoretically this fact is expressed by the isomorphism $L_+^\uparrow \cong \mathrm{SL}(2,\mathbf{C})/\mathbf{Z}_2$. Taking into account that the group $\mathrm{SL}(2,\mathbf{C})$ is simply connected, this means that $\mathrm{SL}(2,\mathbf{C})$ is the *universal covering group* of $L_+^\uparrow$.