Exercise 4.1 *Non-relativistic propagator reloaded* (2 points)

The retarded and advanced propagators $G_{\mathrm{ret/adv}}$ for a Schrödinger wave function $\psi(x) = \psi(t, \mathbf{x})$ with $x^\mu = (t \equiv x_0, \mathbf{x})$ are defined by

$$\begin{aligned} \theta(x_0 - y_0)\,\psi(x) &= +i \int \mathrm{d}^3 y\, G_{\mathrm{ret}}(x-y)\,\psi(y), \\ \theta(y_0 - x_0)\,\psi(x) &= -i \int \mathrm{d}^3 y\, G_{\mathrm{adv}}(x-y)\,\psi(y). \end{aligned}$$

a) Derive the condition

$$\left(i\partial_t + \frac{1}{2m}\triangle - V(x) \right) G_{\mathrm{ret/adv}}(x) = \delta(x)$$

from Schrödinger's equation for $\psi(x)$.

b) In Exercise 3.1 you have calculated the Fourier-transformed propagators $\tilde{G}_{0,\mathrm{ret/adv}}(p) = [p_0 - \mathbf{p}^2/(2m)]^{-1}$ with $p^\mu = (p_0, \mathbf{p})$ of the free theory (i.e. for $V(x) = 0$) from the ansatz

$$G_{\mathrm{ret/adv}}(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{-ipx}\, \tilde{G}_{\mathrm{ret/adv}}(p). \qquad (1)$$

Perform now the p_0-integration in Eq. (1), after shifting the pole in p_0 according to $\tilde{G}_{0,\mathrm{ret/adv}}(p) = [p_0 - \mathbf{p}^2/(2m) \pm i\epsilon]^{-1}$ with an infinitesimal $\epsilon > 0$. Which signs of $\pm i\epsilon$ correspond to the retarded and advanced cases?
(Hint: show and use $\theta(\pm\tau) = \dfrac{\mp 1}{2\pi i} \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{\mathrm{e}^{-i\omega\tau}}{\omega \pm i\epsilon}$.)

c) Calculate $G_{0,\mathrm{ret/adv}}(x)$ explicitly upon carrying out the integration over $\mathrm{d}^3 p$ in (1).
(Hint: use the auxiliary integral $\displaystyle\int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{-a(z+b)^2} = \sqrt{\pi/a}$, where $b \in \mathbf{R}$, $a \in \mathbf{C}$, $a \neq 0$, $\mathrm{Re}(a) \geq 0$.)

Please turn over !

Exercise 4.2 *Electromagnetic interaction of charged scalars* (2 points)

Generically the interaction of the electromagnetic field $A^\mu = (\phi, \mathbf{A})$ with charged fields is described by a Lagrangian of the form

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu + \mathcal{L}_0,$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field-strength tensor, $j^\mu = (\rho, \mathbf{j})$ the 4-current density of the charges, and $\mathcal{L}_0$ the Lagrangian for the free propagation of the charges (and if relevant of other interactions among them), i.e. $\mathcal{L}_0$ does not depend on A^μ.

(Comment: in relativistic field theory it is customary to use Lorentz-Heaviside units, which result from the SI units upon setting $\mu_0, \varepsilon_0, c \to 1$.)

a) Verify the homogenous Maxwell equations for the electric field strength $\mathbf{E} = -\nabla\phi - \dot{\mathbf{A}}$ and the magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$ from the definition of $F^{\mu\nu}$.

b) Derive the inhomogeneous Maxwell equations for the field strength in their covariant form $\partial_\mu F^{\mu\nu} = j^\nu$ from the Euler-Lagrange equations for A^μ and bring them into their usual form in terms of $\mathbf{E}$ and $\mathbf{B}$. Verify current conservation $\partial_\mu j^\mu = 0$.

c) Now consider a complex scalar field Φ to describe a spinless particle with electric charge q and mass m, as in Exercise 3.3.b). The free propagation of Φ is described by

$$\mathcal{L}_0(\Phi, \partial\Phi) = (\partial\Phi)^*(\partial\Phi) - m^2\Phi^*\Phi.$$

The electromagnetic interaction between Φ and A^μ is introduced by the "minimal substitution" $\partial^\mu \to D^\mu = \partial^\mu + iqA^\mu$ in $\mathcal{L}_0$, resulting in

$$\mathcal{L}_\Phi(\Phi, \partial\Phi) = \mathcal{L}_0(\Phi, D\Phi) = \mathcal{L}_0(\Phi, \partial\Phi) - j_\mu A^\mu.$$

Derive the explicit form of the current density j^μ.

d) $\mathcal{L}_\Phi$ is invariant under the global transformation $\Phi \to \Phi' = \exp\{-iq\omega\}\Phi$, with ω denoting an arbitrary real number. Derive the Noether current corresponding to this symmetry and compare it with j^μ from above.