Exercises to Advanced Quantum Mechanics — Sheet 5

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Exercise 5.1 *Time-reversal operator* (3 points)

The time-reversal operator $\mathcal{T}$ is antilinear (!) and defined by the following action on position eigenstates:

$$\mathcal{T}|\vec{x}\rangle = |\vec{x}\rangle. \tag{1}$$

Additionally $\mathcal{T}$ inverts the sign of the external time dependence in potentials $V(\hat{\vec{x}},t)$, i.e. $\mathcal{T}\,V(\hat{\vec{x}},t)\mathcal{T}^{-1} = V(\hat{\vec{x}},-t)$. Recall the basic properties of antilinear and antiunitary operators, as e.g. given in Exercise 2.2. (Spin will not be considered in this exercise.)

a) Show that $\mathcal{T}$ is antiunitary and acts on position-space wave functions as $\mathcal{T}\psi(\vec{x},t) = \psi(\vec{x},t)^*$. Derive the operators $\hat{\vec{x}}'$, $\hat{\vec{p}}'$, $\hat{\vec{L}}'$, where $A' = \mathcal{T}\,A\,\mathcal{T}^{-1}$ is the time-reversed version of an operator A. Here $\hat{\vec{L}} = \hat{\vec{x}} \times \hat{\vec{p}}$ is the usual orbital angular momentum of a single particle.

b) Show that $\mathcal{T}\psi(\vec{x},t) = \psi(\vec{x},-t)$ for states obeying $\mathcal{T}|\psi(t=0)\rangle = |\psi(0)\rangle$ upon deriving the relation $\mathcal{T}U(t,0)\mathcal{T}^{-1} = U(-t,0)$ for the time evolution operator $U(t,t_0)$ of a system with a time-dependent Hamiltonian $\hat{H}(t)$ whose external time dependence is symmetric under time revearsal, i.e. $\hat{H}(-t) = \hat{H}(t)$.

c) Derive the time-reversed operators of the electromagnetic potentials and the field strengths $\vec{A}'(\hat{\vec{x}},t)$, $\Phi'(\hat{\vec{x}},t)$, $\vec{E}'(\hat{\vec{x}},t)$, $\vec{B}'(\hat{\vec{x}},t)$, upon analysing Maxwell's equations and the fact that electric charges do not change sign under time reversal.

Exercise 5.2 *Rotation matrices* (2 points + 1 bonus point)

Consider a rotation about the vector $\vec{\theta} = \theta\vec{e}$ in 3-dimensional space, i.e. about an axis $\vec{e}$ ($\vec{e}^{\,2} = 1$) with an angle θ ($0 \le \theta \le \pi$).

a) Show that the 3×3 matrix $R(\vec{\theta})$ for this rotation is given by

$$R(\vec{\theta}) = \cos\theta\,\mathbb{1} + (1-\cos\theta)\,\vec{e}\,\vec{e}^{\,\mathrm{T}} + \sin\theta\,\vec{e}\times\,, \tag{2}$$

upon directly evaluating the exponential series $R(\vec{\theta}) = \exp\{\vec{\theta}\cdot\vec{I}\}$ with $(I_a)_{bc} = -\epsilon_{abc}$.

b) Derive (short!) formulas that deliver θ and the components of $\vec{e}$ directly from the components of the matrix $R(\vec{\theta})$. Use these results to determine θ and $\vec{e}$ for the rotation

$$R(\vec{\theta}) = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}. \tag{3}$$

c) Earn a bonus point for deriving θ and $\vec{e}$ for a general rotation $R(\alpha,\beta,\gamma)$ that is parametrised by the Euler angles α,β,γ as defined in the lecture.

Please turn over!

Exercise 5.3 *Angular momentum eigenstates reloaded* (2 points)

Consider the operators J_a ($a = 1, 2, 3$) of angular momentum, which obey the commutation relations

$$[J_a, J_b] = \mathrm{i}\hbar \sum_c \epsilon_{abc} J_c, \tag{4}$$

and the related operators $\vec{J}^{\,2} = \sum_a J_a J_a$ and $J_\pm = J_1 \pm \mathrm{i}J_2$.

a) Derive all commutators of J_3, $J_\pm$, and $\vec{J}^{\,2}$.

b) Derive all allowed values of the parameters j and m which parametrise the eigenstates $|j, m\rangle$ of $\vec{J}^{\,2}$ and J_3 as follows,

$$\vec{J}^{\,2}|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle, \qquad J_3|j, m\rangle = \hbar m|j, m\rangle. \tag{5}$$

c) Derive the relations

$$J_\pm|j, m\rangle = c^\pm_{jm}|j, m \pm 1\rangle \tag{6}$$

and explicitly determine the constants $c^\pm_{jm}$.

d) Write down the explicit form of the matrices representing J_3, $J_\pm$, and $\vec{J}^{\,2}$ for the three smallest values of j.