Exercises to Advanced Quantum Mechanics — Sheet 13

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Exercise 13.1 *Resonance scattering* (2 bonus points)

Consider the scattering of a particle (mass M) off some potential that produces a narrow resonance (width $\Gamma \ll E_r$) at an energy E_r in the partial wave with quantum number $l = l_0$. In the following we are only interested in energies E close to the resonance ($|E - E_r| \lesssim \Gamma$), where the scattering phase $\delta_{l_0}(E)$ consists of a smoothly varying part, which can be approximated by a constant $\bar{\delta}_{l_0}$, and the resonance part $\Delta\delta^r_{l_0}$ parametrized by

$$\tan \Delta\delta^r_{l_0} = \frac{\Gamma}{2(E_r - E)}.$$

Derive the total resonance cross section $\sigma_{l_0}(E)$ for generic values of $\bar{\delta}_{l_0}$ and sketch the shape of $\sigma_{l_0}(E)$ for $\bar{\delta}_{l_0} = 0, \pi/2, \pi, 3\pi/2$.
Low-energy electron scattering off noble gases shows cross sections that are extremely suppressed at certain energies (Ramsauer–Townsend effect). What do you conclude on $\bar{\delta}_{l_0}$ for those resonances?

Exercise 13.2 *Probability flux in potential scattering* (3 bonus points)

Asymptotically for large $r = |\vec{x}|$, a scattering wave function behaves as

$$\psi(\vec{x}) \underset{r\to\infty}{\widetilde{\quad}} \psi_{\vec{k}}(\vec{x}) + \psi_{\rm sc}(\vec{x}), \qquad \psi_{\vec{k}}(\vec{x}) = \mathrm{e}^{\mathrm{i}\vec{k}\vec{x}}, \quad \psi_{\rm sc}(\vec{x}) = \frac{\mathrm{e}^{ikr}}{r} f_k(\Omega),$$

where $\hbar\vec{k}$ is the momentum of the incoming particle and Ω the solid angle of $\vec{x}$.

a) Prove that the plane wave has the asymptotic behaviour

$$\mathrm{e}^{\mathrm{i}\vec{k}\vec{x}} \underset{r\to\infty}{\widetilde{\quad}} \frac{2\pi}{\mathrm{i}kr} \left[\delta(\Omega - \Omega_{\vec{k}})\mathrm{e}^{ikr} - \delta(\Omega + \Omega_{\vec{k}})\mathrm{e}^{-ikr} \right],$$

where $\Omega_{\vec{k}}$ the solid angle of $\vec{k}$ and the angular delta functions are defined in terms of the corresponding polar and azimuthal angles θ and ϕ as

$$\delta(\Omega - \Omega_{\vec{k}}) = \delta(\cos\theta - \cos\theta_{\vec{k}})\, \delta(\phi - \phi_{\vec{k}}).$$

Hint: Recall Exercises 1.3 and 11.3 and use various properties of the spherical harmonics.

b) Calculate the particle flux through a sphere of a large radius R ($kR \gg 1$) originating from the wave functions $\psi_{\vec{k}}$ and $\psi_{\rm sc}$ alone.

c) Calculate the total particle flux F through the large sphere originating from the full wave functions $\psi = \psi_{\vec{k}} + \psi_{\rm sc}$. Which relation do you get by demanding $F = 0$?