Exercise 9.1 *Anomalous Zeeman effect* (4 points)

We consider a one-electron atom (relative nucleus charge Z) in a weak homogeneous magnetic field of the strength $\vec{B} = B\vec{e}_3$ and consider the interaction Hamiltonian $\hat{H}_B$ of the atom with the magnetic field as perturbation,

$$\hat{H}_B \;=\; \frac{\mu_\mathrm{B}}{\hbar}\left(\hat{\vec{L}} + g_\mathrm{e}\vec{S}\right)\cdot\vec{B},$$

where $\mu_\mathrm{B} = \frac{e\hbar}{2m_\mathrm{e}}$ is Bohr's magneton and $g_\mathrm{e} = 2.0023...$ the gyromagnetic ratio of the electron. The unperturbed Hamiltonian reads

$$\hat{H} \;=\; \frac{\hat{\vec{p}}^{\,2}}{2m_\mathrm{e}} - \frac{Ze^2}{4\pi\varepsilon_0\hat{r}} + \hat{H}_\mathrm{FS},$$

where $\hat{H}_\mathrm{FS}$ is the Hamiltonian responsible for the atomic fine structure (as discussed in the lecture). The unperturbed energy eigenstates $|nljm\rangle$ are eigenstates of the operators $\hat{\vec{L}}^{\,2}$, $\vec{S}^{\,2}$, $\vec{J}^{\,2}$, $\hat{J}_3$, with the usual parametrization of their eigenvalues by the numbers $l = 0, 1, \ldots$; $s = \frac{1}{2}$; $j = l \pm \frac{1}{2}$; $m = -j, \ldots, j$, respectively. The unperturbed energy levels are

$$E_{nj} \;=\; E_n\left[1 + \frac{(Z\alpha)^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right], \qquad E_n \;=\; -\frac{Z^2}{n^2}E_\mathrm{R}, \quad n = 1, 2, \ldots,$$

where α is the fine-structure constant and E_R Rydberg's energy.

a) Derive the energy shift ΔE_B of the energy levels induced by $\hat{H}_B$, using 1st-order perturbation theory and the relation

$$\begin{aligned}|j = l \pm \tfrac{1}{2}, m\rangle \quad = \quad & \pm\sqrt{\frac{l \pm m + \frac{1}{2}}{2l+1}}\left|m_l = m - \frac{1}{2}, m_s = \frac{1}{2}\right\rangle \\ & + \sqrt{\frac{l \mp m + \frac{1}{2}}{2l+1}}\left|m_l = m + \frac{1}{2}, m_s = -\frac{1}{2}\right\rangle\end{aligned}$$

between the eigenstates $|j, m\rangle$ of $\vec{J}^{\,2}$, $\hat{J}_3$ and the eigenstates $|m_l, m_s\rangle$ of $\hat{L}_3$ and S_3.

b) To prepare an alternative derivation, first show that

$$\hbar^2 j(j+1)\langle j, m|\vec{V}|j, m\rangle \;=\; \langle j, m|(\vec{V}\cdot\vec{J})\vec{J}|j, m\rangle \tag{1}$$

for any vector operator $\vec{V}$, upon exploiting the identity

$$[\vec{J}^{\,2}, [\vec{J}^{\,2}, \vec{V}]] = 2\hbar^2(\vec{J}^{\,2}\vec{V} + \vec{V}\vec{J}^{\,2}) - 4\hbar^2(\vec{V}\cdot\vec{J})\vec{J}.$$

c) Derive ΔE_B using relation (1).

d) Sketch the energy levels $E_{nj} + \Delta E_B$ as functions of B for all states emerging from the unperturbed states $nl_j = 1s_{1/2}, 2s_{1/2}, 2p_{1/2}, 2p_{3/2}$.

Please turn over!

Exercise 9.2 *Linear Stark effect* (3 points)

We consider a one-electron atom (relative nucleus charge Z) in a homogeneous electric field of the strength $\vec{\mathcal{E}} = \mathcal{E}\vec{e}_3$ and consider the interaction Hamiltonian $\hat{H}_E$ of the atom with the electric field as perturbation,

$$\hat{H}_E \;=\; e\mathcal{E}\hat{x}_3.$$

For sufficiently strong electric fields, spin effects, atomic fine structure, and other corrections can be neglected in a first approximation, so that the unperturbed Hamiltonian reads

$$\hat{H} \;=\; \frac{\hat{\vec{p}}^{\,2}}{2m_\mathrm{e}} - \frac{Ze^2}{4\pi\varepsilon_0\hat{r}}.$$

The unperturbed energy eigenstates $|nlm_l\rangle$ are eigenstates of the operators $\hat{\vec{L}}^{\,2}$ and $\hat{L}_3$, with the usual parametrization of their eigenvalues by the numbers $l = 0, 1, \ldots$ and $m_l = -l, \ldots, l$, respectively.

a) Calculate the first-order energy shift ΔE_{nlm_l} of a general s state ($l = 0$).

b) Calculate the first-order energy shifts ΔE_{nlm_l} for the first excited states ($n = 2$).

c) Calculate the second-order energy shift $\Delta E^{(2)}_{100}$ of the ground state approximately upon replacing $E_{n>1}$ by E_2 in the underlying formula.

[Hint: Using $\langle\hat{x}_j^2\rangle_{100} = \langle\hat{r}^2\rangle_{100}/3 = a_\mathrm{B}^2/Z^2$ saves you the radial integral.]

Exercise 9.3 *Linear potential and variational method* (2 points)

As in Exercise 8.3, consider a particle with mass m in a one-dimensional potential $V(x) = \varepsilon|x|$ with $\varepsilon > 0$. As trial function for variations we use $\phi_0(x) = \exp(-\alpha x^2)$ and $\phi_1(x) = x\exp(-\beta x^2)$ in the following, where α and β are independent free real parameters and $\alpha, \beta > 0$.

a) Determine an approximation for the ground state energy E_0 upon minimizing the energy expectation value for a wave function $\psi(x)$ formed by an optimal linear combination of the trial functions. Compare the result with the one obtained in Exercise 8.3.

b) Determine an approximation for the energy E_1 of the first excited state similar to the procedure for the ground state. Again, compare the result with the one obtained in Exercise 8.3.

Hint: Symmetry arguments simplify the actual calculations.