Exercise 12.1 *Charged-particle scattering off a charge distribution* (4 points)

We consider the scattering of a point-like, spinless particle with electric charge $Q_1 e$ and mass M off a smoothly distributed charge, whose spherically symmetric charge density $\rho(r)$ is centered at $r = 0$.

a) Show that the electric potential $\Phi(r)$ of the charge distribution can be written as

$$\Phi(r) \;=\; \frac{1}{\varepsilon_0 r}\int_0^r \mathrm{d}\bar{r}\,\bar{r}^2\rho(\bar{r}) - \frac{1}{\varepsilon_0}\int_\infty^r \mathrm{d}\bar{r}\,\bar{r}\rho(\bar{r}).$$

b) Assuming first that the total charge $Q_2 e = 4\pi\int_0^\infty \mathrm{d}\bar{r}\,\bar{r}^2\rho(\bar{r})$ of the distribution vanishes, calculate the scattering amplitude $f_k(\Omega)$ in Born approximation. Express the result in terms of the *form factor* $F(q)$ of the charge distribution, which is defined by

$$F(q) \;=\; \int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}\vec{q}\vec{x}}\,\rho(r)/e, \qquad r = |\vec{x}|, \quad q = |\vec{q}|.$$

You should find that $f_k(\Omega) = \text{const.} \times F(q)/q^2$ with $q = 2k\sin(\theta/2)$.

c) Derive the corresponding differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ and identify its relation to the Rutherford cross section. Argue why the result is valid if for $Q_2 \neq 0$ as well. What is the limit of $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ for $ka \ll 1$ if $Q_2 \neq 0$ and a is a sphere containing the entire charge distribution?

d) Derive the differential Born cross section for a homogeneous charge distribution contained in a sphere of radius a.

Rutherford scattered α particles with a velocity of $v = 1.5 \cdot 10^7\,\mathrm{m/s}$ off gold nuclei and did not find a significant deviation from the cross section for a point-like nucleus charge. Derive a rough upper bound on a from this result, interpreting a as the nucleus radius in the model of a homogeneous charge distribution in the nucleus.

Please turn over!

Exercise 12.2 *Scattering off hard spheres* (3 points)

We consider the scattering of particles of mass M at the potential

$$V(r) = \begin{cases} 0, & r > a, \\ \infty, & r < a \end{cases}$$

of a "hard sphere".

a) Determine $\tan\delta_l(k)$ of the scattering phases $\delta_l(k)$ from the exact solution of the Schrödinger equation.

b) Calculate the leading behaviour of the differential and total cross sections in the low-energy limit $ka \ll 1$, keeping terms up to order k^2.

c) Show that the total cross section at high energies ($ka \gg 1$) is approximately given by $2\pi a^2$. Compare this result with the one obtained in classical mechanics.

   Hint: As argued in the lecture, only partial waves with $l \lesssim ka$ are relevant. Moreover, in good approximation the asymptotics of $j_l(\rho)$ and $n_l(\rho)$ for large ρ can be used.