Exercise 12 *Chiral supermultiplet of $N = 2$ supersymmetry* (12 points)

The part of the $N = 2$ super Poincaré algebra that will be relevant in the following is given by

$$\{Q_{a,r}, \bar{Q}_{\dot{b},s}\} = 2\delta_{rs}\bar{\sigma}^{\mu}_{a\dot{b}}P_{\mu}, \qquad \{Q_{a,r}, Q_{b,s}\} = \epsilon_{ab}\epsilon_{rs}Z, \qquad \{\bar{Q}_{\dot{a},r}, \bar{Q}_{\dot{b},s}\} = \epsilon_{\dot{a}\dot{b}}\epsilon_{rs}Z^*, \tag{1}$$

$$[J^k, Q_{a,r}] = -\frac{1}{2}\sigma^k_{ab}Q_{b,r}, \qquad [J^k, \bar{Q}_{\dot{a},r}] = +\tfrac{1}{2}(\sigma^k)^*_{\dot{a}\dot{b}}\bar{Q}_{\dot{b},r}, \tag{2}$$

where $r, s = 1, \ldots, N = 2$. The angular momentum operator is denoted by J^k $(k = 1, 2, 3)$, σ^k are the usual Pauli matrices. The central charge Z commutes with all symmetry operators and is proportional to unity in irreducible representations, which we shall consider in the following.

a) Show that one can assume, without loss of generality, Z to be non-negative and real.
(Hint: Redefine the SUSY generators appropriately!)

b) Consider states of massive particles in their rest frame, i.e. states with the momentum eigenvalue $p^\mu = (M, \mathbf{0})$. Using these states evaluate the anticommutators of the operators

$$a_1 = \frac{1}{\sqrt{2}}\left(Q_{1,1} + \bar{Q}_{\dot{2},2}\right), \qquad a_2 = \frac{1}{\sqrt{2}}\left(Q_{2,1} - \bar{Q}_{\dot{1},2}\right), \tag{3}$$

$$b_1 = \frac{1}{\sqrt{2}}\left(Q_{1,1} - \bar{Q}_{\dot{2},2}\right), \qquad b_2 = \frac{1}{\sqrt{2}}\left(Q_{2,1} + \bar{Q}_{\dot{1},2}\right), \tag{4}$$

as well as the anticommutators of the adjoint operators, and the anticommutators of the operators with their adjoints.

c) Using these relations show that $Z \leq 2M$. Which consequence has $Z = 2M$ for the representation of the operators? How should a_r and b_r normalized so that they define creation and annihilation operators?

d) Evaluate the commutators of the operators a_r, $a_r^\dagger$, b_r, $b_r^\dagger$ with the angular momentum operators J^3 and $J^\pm = J^1 \pm iJ^2$.

e) The "chiral supermultiplet" is defined via the Clifford vacuum $|\Omega\rangle$ which has spin zero and satisfies $a_r|\Omega\rangle = b_r|\Omega\rangle = 0$. Write down all states that are generated by repeatedly applying $a_r^\dagger$ and $b_r^\dagger$ to the Clifford vacuum. How are these states normalized? What is their eigenvalue of J_3?

f) Group the states from e) together according to their spin eigenvalue.