Exercises on Supersymmetry **Sheet 2**

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Exercise 4 *Relations for Grassmann-valued Weyl spinors* (4 points)

Let ξ and η be Grassmann-valued Weyl spinors.

a) Derive the following relations, using the Schouten identity of the antisymmetric metric ϵ:

$$\xi^a \eta^b = \xi^b \eta^a - \epsilon^{ab}(\xi\eta), \quad \xi^a \xi^b = -\frac{1}{2}\epsilon^{ab}(\xi\xi), \tag{1}$$

$$\bar{\xi}^{\dot{a}} \bar{\eta}^{\dot{b}} = \bar{\xi}^{\dot{b}} \bar{\eta}^{\dot{a}} + \epsilon^{\dot{a}\dot{b}}(\bar{\xi}\bar{\eta}), \quad \bar{\xi}^{\dot{a}} \bar{\xi}^{\dot{b}} = +\frac{1}{2}\epsilon^{\dot{a}\dot{b}}(\bar{\xi}\bar{\xi}). \tag{2}$$

b) Prove that the objects

$$\bar{\xi}\sigma^\mu \eta \equiv \bar{\xi}_{\dot{a}} \sigma^{\mu,\dot{a}b} \eta_b, \qquad \xi\bar{\sigma}^\mu \bar{\eta} \equiv \xi^a \bar{\sigma}^\mu_{a\dot{b}} \bar{\eta}^{\dot{b}} \tag{3}$$

transform as four-vectors and fulfill the following relations,

$$\xi\bar{\sigma}^\mu\bar{\eta} = -\bar{\eta}\sigma^\mu\xi, \tag{4}$$

$$\xi^a \bar{\eta}^{\dot{b}} = \tfrac{1}{2}(\xi\bar{\sigma}^\mu\bar{\eta})\bar{\sigma}_\mu^{a\dot{b}}, \tag{5}$$

$$(\xi\bar{\sigma}^\mu\bar{\eta})(\xi\bar{\sigma}^\nu\bar{\eta}) = \tfrac{1}{2}(\xi\xi)(\bar{\eta}\bar{\eta})g^{\mu\nu}. \tag{6}$$

Exercise 5 *Commuting Weyl spinors* (2 point)

How do the relations in Exercise 4 change if the anticommutating spinors ξ and η are replaced by commuting spinors x and y? In particular, which symmetry has the spinor-product $\langle xy \rangle \equiv x_a y^a$? Furthermore, compare the definition $\langle \bar{x}\bar{y} \rangle \equiv \langle xy \rangle^*$ with the corresponding one for anticommutating spinors. How does it differ?

Exercise 6 *Representation of four-vectors by (commuting) Weyl spinors* (3 points)

Let k^μ be a real four-vector which is mapped onto a 2×2-matrix $K_{\dot{a}b}$,

$$(K_{\dot{a}b}) \equiv (k^\mu \sigma_{\mu,\dot{a}b}) = \begin{pmatrix} k^0 + k^3 & k^1 + ik^2 \\ k^1 - ik^2 & k^0 - k^3 \end{pmatrix}. \tag{7}$$

a) Show that a matrix with the property $K_{\dot{a}b} = \bar{k}_{\dot{a}} k_b$, where k_a is a commuting Weyl spinor, corresponds to a light-like vector k^μ ($k^2 = 0$).
(Hint: Examine the determinant of K.)

b) Show the reversed case, i.e. that every light-like vector can be represented by $\bar{k}_{\dot{a}} k_b$. Construct k_a with the components of k^μ in polar coordinates,

$$(k^\mu) = k^0 \,(1, \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)\,. \tag{8}$$

Which properties of k_a are unambiguously fixed, which can be freely chosen?

c) How is the Minkowski product $p \cdot k$ of two light-like vectors p, k expressed in terms of the spinor product $\langle \ldots \rangle$ of the corresponding spinors?

Exercise 7 *Wave function of massless Dirac fermions* (4 points)

Show that in the representation

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\bar{\sigma}^\mu_{a\dot{b}}) \\ (\sigma^{\mu,\dot{a}b}) & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

both wave functions

$$\Psi_{\mathrm{R}} = \begin{pmatrix} k_a \\ 0 \end{pmatrix}, \qquad \Psi_{\mathrm{L}} = \begin{pmatrix} 0 \\ \bar{k}^{\dot{a}} \end{pmatrix} \tag{10}$$

are solutions of the Dirac equation for a massless fermion with momentum k^μ and are suitably normalized, i.e. $\Psi^\dagger\Psi = 2k^0$. What are the adjoint spinors $\overline{\Psi}_{\mathrm{R/L}} \equiv \Psi^\dagger_{\mathrm{R/L}}\gamma_0$?