Exercise 5.1 (3 points) *Integration over Grassmann variables*

We consider a Grassmann algebra with the $2N$ generating elements y_k, y_k^* $(k = 1, \ldots, N)$.

a) The generators y_k will be replaced by new generators z_k with the mapping $z_i = a_{ik} y_k$, where a_{ik} denote the coefficients of a complex matrix A. Show that the corresponding differentials transform as $dz_i = \tilde{a}_{ik} dy_k$, where $\tilde{a}_{ik}$ are the coefficients of the matrix that is the transposed inverse matrix of A.

b) Using a), derive

$$\int dz_1 \cdots \int dz_n \, F(\mathbf{z}) \;=\; (\det A)^{-1} \int dy_1 \cdots \int dy_n \, F(\mathbf{z}(\mathbf{y})).$$

c) With the help of b) show that the Gaussian integral is given by

$$\int dy_1 \cdots \int dy_n \int dy_n^* \cdots \int dy_1^* \, \exp\{y_i^* a_{ik} y_k\} \;=\; \det A.$$

Exercise 5.2 (4 points) *Generating functional of free Dirac fields*

The generating functional of free Dirac fields ψ, $\overline{\psi}$ is given by the functional integral

$$Z_{\psi,0}[\eta,\overline{\eta}] = N \int \mathcal{D}\psi \int \mathcal{D}\overline{\psi} \, \exp\left\{ i \int d^4x \, \Big[\overline{\psi}(x)\,(i\partial\!\!\!/ - m)\,\psi(x) + \overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x) \Big] \right\}$$

with the normalization $Z_{\psi,0}[0,0] = 1$. The source fields η, $\overline{\eta}$ are of Grassmann-type.

a) Analogous to the procedure for scalar fields, calculate the functional $Z_{\psi,0}[\eta,\overline{\eta}]$ as

$$Z_{\psi,0}[\eta,\overline{\eta}] = \exp\left\{ \int d^4x \int d^4x' \, i\overline{\eta}(x) \, iS_F(x-x') \, i\eta(x') \right\},$$

where $iS_F(x)$ is defined by $(i\partial\!\!\!/ - m)\; S_F(x) = \delta(x)$.

b) Starting from the generating functional $Z_{\psi,0}[\eta,\overline{\eta}]$, derive the fermion propagator

$$G_0^{\overline{\psi}\psi}(x_1,x_2) = \frac{\delta}{i\delta\eta(x_2)} \frac{\delta}{i\delta\overline{\eta}(x_1)} \, Z_{\psi,0}[\eta,\overline{\eta}]\Big|_{\eta,\overline{\eta}=0}.$$

c) Solve the differential equation for $S_F(x)$ in momentum space and derive an explicit form of the Fourier representation of the propagator

$$G_0^{\overline{\psi}\psi}(x_1,x_2) = \int \frac{d^4q}{(2\pi)^4} \, \mathrm{e}^{-iq(x_1-x_2)} \, \tilde{G}_0^{\overline{\psi}\psi}(q).$$

Please turn over !

Exercise 5.3 (3 points) *Green's function for the elementary quark–gluon interaction*

The generating functional of QCD can be written as

$$\begin{aligned} Z[J^a_\mu,\eta,\overline{\eta}] &= N\,\exp\left\{i\int d^4y\,\mathcal{L}_I\left(A^a_\mu(y)\to\frac{\delta}{i\delta J^{\mu,a}(y)},\overline{\psi}(y)\to-\frac{\delta}{i\delta\eta(y)},\psi(y)\to\frac{\delta}{i\delta\overline{\eta}(y)}\right)\right\} \\ &\quad\times Z_{A,0}[J^a_\mu]\,Z_{\psi,0}[\eta,\overline{\eta}], \\ Z[0,0,0] &= 1, \\ Z_{A,0}[J^a_\mu] &= \exp\left\{\tfrac{1}{2}\int d^4x\int d^4x'\,iJ^{\mu,a}(x)\,G^{AA,ab}_{0,\mu\nu}(x,x')\,iJ^{\nu,b}(x')\right\}, \\ Z_{\psi,0}[\eta,\overline{\eta}] &= \exp\left\{\int d^4x\int d^4x'\,i\overline{\eta}(x)\,G^{\overline{\psi}\psi}_0(x,x')\,i\eta(x')\right\}, \end{aligned}$$

where the Lagrangian for the quark–gluon interaction is given by $\mathcal{L}_I = -g\overline{\psi}A^a_\mu T^a\gamma^\mu\psi$. Calculate the Green's function

$$G^{A\overline{\psi}\psi,a}_\mu(x_1,x_2,x_3) = \frac{\delta}{i\delta J^{\mu,a}(x_1)}\frac{\delta}{i\delta\eta(x_3)}\frac{\delta}{i\delta\overline{\eta}(x_2)}\,Z[J^a_\mu,\eta,\overline{\eta}]\Big|_{J^a_\mu,\eta,\overline{\eta}\to 0}$$

in lowest-order pertubation theory and, after transforming into momentum space, verify that the corresponding amputated Green's function agrees with the Feynman rule $-igT^a\gamma_\mu$.