Exercise 2.1 (5 points) *R ratio for e^+e^- annihilation*

The so-called R ratio is defined as

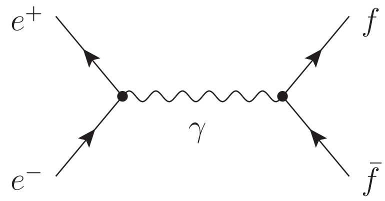
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

where $\sigma(e^+e^- \to \text{hadrons})$ denotes the total production cross section for hadronic final states. In the quark-parton model, in first-order approximation, this cross section can be calculated from the production cross section of quark-antiquark pairs:

$$\sigma(e^+e^- \to \text{hadrons}) \approx \sum_q \sigma(e^-e^+ \to q\bar{q})$$

where, at a centre-of-mass energy E_{CM}, the quark species q with mass $m_q < E_{CM}/2$ contribute.

For the calculation of the R ratio in QED, the process $e^+(p_+)+e^-(p_-) \to f(k_1)+\bar{f}(k_2)$ where f is a fermion with charge Q_f, i.e. either a muon ($Q_\mu = -1$), a $u/c/t$ quark ($Q_f = 2/3$), or a $d/s/b$ quark ($Q_f = -1/3$). The masses of the fermions can be neglected. In QED, in Born approximation only the following diagram contributes to this reaction.



The particle momenta are given in the centre-of-mass system as

$$p^\mu_\pm = E(1, 0, 0, \pm 1), \qquad k^\mu_{1,2} = E(1, \pm\sin\theta\cos\varphi, \pm\sin\theta\sin\varphi, \pm\cos\theta) \ ,$$

where $E = E_{CM}/2$ is the beam energy in the centre-of-mass system and θ, φ denote the usual angles in polar coordinates for the outgoing momentum of f.

a) Calculate the spin-averaged squared transition matrix element $\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{pol.} |\mathcal{M}|^2$ for the process $\sigma(e^+e^- \to \mu^-\mu^+)$

b) Determine the 2-particle phase space upon using the general n-particle phase space measure

$$\mathrm{dPS}_n = \frac{1}{(2\pi)^{3n-4}} \prod_{f=1}^{n} \frac{\mathrm{d}^3 p_f}{2p^0_f}\, \delta^{(4)}\left(\sum_{j=1}^{n} p_j - p_{i,1} - p_{i,2}\right) \ , \qquad (1)$$

where n is the number of particles in the final state and $p_{i,1/2}$ are the two incoming momenta.

c) Determine the differential cross section $\mathrm{d}\sigma/\mathrm{d}\cos\theta$ as well as the total cross section σ.

d) Calculate the total cross section for the processs $e^+e^- \to q\bar{q}$ analogous to a). Note that, in this case, you have to sum over the number N_C of the colour degrees of freedom of the quarks.

e) Calculate the R ratio for each of the centre-of-mass energies $E_{CM} < 2.5$ GeV, 4 GeV $< E_{CM} <$ 9 GeV and 10 GeV $< E_{CM} <$ 90 GeV. The quark masses are given as $m_u \approx 2.5$ MeV, $m_d \approx 5$ MeV, $m_s \approx 100$ MeV, $m_c \approx 1.3$ GeV, $m_b \approx 4.7$ GeV, $m_t \approx 173$ GeV. Compare the results with the experimental findings, which you can retrieve from http://pdg.lbl.gov/2017/hadronic-xsections (at the bottom), and extract the value of N_C. Do you observe a systematic deviation from the expectation?

Exercise 2.2 (5 points) *SU(3) identities*

In this exercise the generators of $T^a = \lambda^a/2$ of SU(3) and their Lie algebra $[T^a, T^b] = if^{abc}T^c$ given in exercise 1.1) will be reconsidered with generators fulfilling the standard normalization $\mathrm{Tr}(T^aT^b) = T_F\delta_{ab}$ with $T_F = \frac{1}{2}$.

a) Prove the identity

$$T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right).$$

Make use of the fact that the generators T^a and the unit matrix form a basis of 3×3 matrices.

b) As shown in exercise 1.1.a) T^aT^a is a Casimir operator. According to the Schur's lemma the Casimir operator is proportional to the identity, i.e.

$$(T^aT^a)_{ij} = C_F\delta_{ij}.$$

holds. Calculate the constant C_F using the result of a).

c) Reduce the combinations $T^aT^bT^a$ and $T^aT^bT^cT^a$ to linear combinations of single generators T^a and the unit matrix.

d) The generators of the *adjoint representation* are given by $(T^a_{\mathrm{adj}})_{bc} = -if^{abc}$. Show that these generators fulfill the Lie algebra $[T^a_{\mathrm{adj}}, T^b_{\mathrm{adj}}] = if^{abc}T^c_{\mathrm{adj}}$ applying the Jacobi identity.

e) Calculate the Casimir operator

$$\left(T^a_{\mathrm{adj}}T^a_{\mathrm{adj}}\right)_{bc} = C_A\delta_{bc}$$

in the adjoint representation.
(Hint: Calculate $\mathrm{Tr}\left\{[T^a, T^b][T^a, T^c]\right\}$ in two different ways.)