

Exercise 9.1 (1.5 points) *Vertex functions of scalar QED*

The electromagnetic interaction of spin-0 particles S^\pm with mass M (field variable ϕ for particles S^- with charge Q) is described by $\mathcal{L}_\phi = (D_\mu\phi)^\dagger(D^\mu\phi) - M^2\phi^\dagger\phi$ with $D_\mu = \partial_\mu + iQeA_\mu$. This yields the Feynman rules of *scalar QED* given on the back side.

- a) Calculate the S -boson self-energy $\Sigma^{SS}(p^2)$ in one-loop approximation in Feynman gauge. The self-energy is defined by the two-point vertex function as follows:

$$\Gamma^{S^+S^-}(-p, p) = i \left[p^2 - M^2 + \Sigma^{SS}(p^2) \right].$$

- b) The three-point vertex function

$$\Gamma_\mu^{AS^+S^-}(k, p_+, p_-) = -iQe \left[(p_- - p_+)_\mu + \Lambda_\mu(-p_+, p_-) \right]$$

fulfills the Ward identity

$$k^\mu \Gamma_\mu^{AS^+S^-}(k, p_+, p_-) = Qe \left[\Gamma^{S^+S^-}(-p_-, p_-) - \Gamma^{S^+S^-}(p_+, -p_+) \right].$$

Using this identity and the fact that the UV-divergent part of $\Gamma_\mu^{AS^+S^-}$ is proportional to the lowest order $\Gamma_{0,\mu}^{AS^+S^-}$, derive the UV-divergent one-loop contribution to the correction $\Lambda_\mu(-p_+, p_-)$.

- c) Analogous to QED with Dirac fermions, calculate the photon self-energy $\Sigma_{\mu\nu}^{AA}(k)$ in one-loop approximation.

Exercise 9.2 (1.5 points) *Renormalization of scalar QED*

- a) Apply the renormalization transformation

$$e \rightarrow e(1 + \delta Z_e), \quad M^2 \rightarrow M^2 + \delta M^2, \quad A^\mu \rightarrow A^\mu \left(1 + \frac{1}{2} \delta Z_A \right), \quad \phi \rightarrow \phi \left(1 + \frac{1}{2} \delta Z_\phi \right)$$

in one-loop approximation to the Lagrangian \mathcal{L}_ϕ given in Exercise 9.1 and determine the generated counterterms (solution on the back side). Which counterterm contributions result for the renormalized vertex functions $\hat{\Gamma}^{S^+S^-}(-p, p)$, $\hat{\Gamma}_\mu^{AS^+S^-}(k, p_+, p_-)$ and $\hat{\Gamma}_{\mu\nu}^{AAS^+S^-}(k_1, k_2, p_+, p_-)$?

- b) Determine δM^2 from the mass-renormalization condition

$$\hat{\Gamma}^{S^+S^-}(-p, p) \Big|_{p^2=M^2} = 0.$$

- c) Determine δZ_ϕ from the renormalization condition

$$\text{Res}_{p^2=M^2} \left\{ \hat{G}^{S^+S^-}(-p, p) \right\} = i$$

for the residue of the S propagator.

- d) Determine δZ_e from the charge-renormalization condition (Thomson limit)

$$\hat{\Gamma}_\mu^{AS^+S^-}(k=0, -p, p) \Big|_{p^2=M^2} = -2iQep_\mu.$$

Using the Ward identity given in Exercise 9.1 b) show that $\delta Z_e = -\frac{1}{2}\delta Z_A$.

Feynman rules for charged spin-0 particles S^\pm :

Vertices:

$$\begin{array}{cc}
 \begin{array}{c} S^+ \text{---} p_+ \\ \text{---} \bullet \\ \text{---} S^- \text{---} p_- \end{array} \begin{array}{c} \text{---} \mu \gamma \end{array} & -iQe(p_- - p_+)_\mu \\
 & \\
 \begin{array}{c} S^+ \text{---} \bullet \\ \text{---} \bullet \\ \text{---} S^- \end{array} \begin{array}{c} \text{---} \mu \gamma \\ \text{---} \nu \gamma \end{array} & 2iQ^2e^2g_{\mu\nu}
 \end{array}$$

Propagator and external lines:

$$\begin{array}{cc}
 S^+ \bullet \text{---} \text{---} \bullet S^- & \frac{i}{p^2 - M^2} \\
 \text{---} p & \\
 \bullet \text{---} \text{---} S^+ & 1 \\
 \bullet \text{---} \text{---} S^- & 1
 \end{array}$$

Counterterms:

$$\begin{array}{cc}
 S^+ \text{---} \text{---} \text{---} S^- & i(p^2 - M^2)\delta Z_\phi - i\delta M^2 \\
 \text{---} p & \\
 \begin{array}{c} S^+ \text{---} p_+ \\ \text{---} \bullet \\ \text{---} S^- \text{---} p_- \end{array} \begin{array}{c} \text{---} \mu \gamma \end{array} & -iQe(p_- - p_+)_\mu \left(\delta Z_e + \delta Z_\phi + \frac{1}{2}\delta Z_A \right) \\
 & \\
 \begin{array}{c} S^+ \text{---} \bullet \\ \text{---} \bullet \\ \text{---} S^- \end{array} \begin{array}{c} \text{---} \mu \gamma \\ \text{---} \nu \gamma \end{array} & 2iQ^2e^2g_{\mu\nu} \left(2\delta Z_e + \delta Z_\phi + \delta Z_A \right)
 \end{array}$$

The fields S^\pm are always assumed to be incoming at the vertex. Outgoing fields S^\pm correspond to the incoming fields S^\mp with reversed momenta. The Feynman rules for external and internal photons are the same as in QED with Dirac fermions.